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A NOTE ON ORDER ONE CYCLOTOMIC POLYNOMIALS

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ABSTRACT. It is a well known fact that, if p is an odd prime, then the p th-elementary cyclotomic polynomial $\Phi_p(x)$ has an associated p -Eisenstein polynomial $\widehat{\Phi}_p(x)$. We extend this construction and show that, every order one elementary cyclotomic polynomial $\Phi_{2^s p^t}(x)$ has an associated p -Eisenstein polynomial $\widehat{\Phi}_{2^s p^t}(x)$. In addition, for each $\Phi_{2^s p^t}(x)$, we investigate the divisibility (with respect to the prime p) of the coefficients of $\widehat{\Phi}_{2^s p^t}(x)$. We also establish analogous results for order one Carlitz cyclotomic polynomials over $\mathbb{F}_q[T]$.

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Key words: Eisenstein and Carlitz cyclotomic polynomials, logarithmic and Mahler heights.

Introduction. This article is motivated by the rich analogy between number fields and function fields, in particular the base rings $\mathbb{Z} \leftrightarrow \mathbb{Q}$ and $\mathbb{F}_q[T] \leftrightarrow \mathbb{F}_q(T)$. We explore the Eisenstein property exhibited by order one cyclotomic polynomials in the two rings¹ and present our findings in Sections 1 and 2. Section 1 deals with the theory of elementary cyclotomic polynomials over $\mathbb{Q}$. In this section, we construct p -Eisenstein polynomials associated to order one elementary cyclotomic polynomials and then study the divisibility of their coefficients with respect to p . Section 2, is devoted to establishing analogous results in $\mathbb{F}_q(T)$.

Let us fix our notation right away; throughout the article, n, s, t shall denote positive integers, p_1, p_2 -odd primes in $\mathbb{Z}$; a, b, m are monic polynomials in $A := \mathbb{F}_q[T]$ and P, Q shall be monic irreducible polynomials in A .

1. Elementary cyclotomic polynomials.

1.1. Properties of elementary cyclotomic polynomials.

¹that is to say, the order one elementary cyclotomic polynomials over $\mathbb{Z}$ as discussed later in Section 1 on the one hand and and their $\mathbb{F}_q[T]$ -analogues, the order one Carlitz cyclotomic polynomials in Section 2.

DEFINITION 1.1. The n th-cyclotomic polynomial over $\mathbb{Q}$ is the polynomial

$$\Phi_n(x) := \prod_{\zeta \in \mu_n: \text{primitive}} (x - \zeta),$$

where μ_n is the set of all n th-roots of unity in $\mathbb{C}$.

It is not hard to show that $\Phi_n(x)$ is monic, with integer coefficients and degree $\varphi(n)$. For $n > 2$, $\Phi_n(x)$ is irreducible over $\mathbb{Q}$. Equivalently, $\Phi_n(x)$ is the minimal polynomial for the primitive n th roots of unity.

PROPOSITION 1.2. ([5], Lemma 1.2)

$$x^n - 1 = \prod_{d|n} \Phi_d(x), \quad (1)$$

where d runs over positive divisors of n .

Proposition 1.2 is sometimes used as a recursive formula for computing $\Phi_n(x)$. This definition works well for small values of n , but in practice, it is a very expensive computation. Applying the (multiplicative form of the) Möbius inversion formula to equation (1), we get the following formula

$$\Phi_n(x) = \prod_{d|n} (x^{\frac{n}{d}} - 1)^{\mu(d)}, \quad (2)$$

where μ is the Möbius function. Recall, the Möbius function is defined as $\mu(n) = (-1)^s$, if n is square free with s distinct primes, and 0 otherwise. The product in equation (2) is also over positive divisors of n .

PROPOSITION 1.3. ([5], Lemma 1.3) *Let n, s be positive integers and p be a prime.*

(a)

$$\Phi_{np^s}(x) = \begin{cases} \Phi_n(x^{p^s}), & (n, p) \neq 1 \\ \Phi_{np}(x^{p^{s-1}}), & (n, p) = 1. \end{cases}$$

(b) *If n is odd, then $\Phi_{2n}(x) = \Phi_n(-x)$.*

(c) *If n_0 be the square free part of n , then $\Phi_n(x) = \Phi_{n_0}(x^{\frac{n}{n_0}})$.*

1.2. Coefficients of order one cyclotomic polynomials. We define the *order* of $\Phi_n(x)$ to be the number of distinct odd prime factors of n . To explain what a p -Eisenstein polynomial associated to an order one cyclotomic polynomial is, we need Definition 1.4.

DEFINITION 1.4. Let $f = x^s + a_{s-1}x^{s-1} + \cdots + a_0 \in \mathbb{Z}[x]$. f is an Eisenstein polynomial (p -Eisenstein form) if there exists prime p such that (i) p divides a_i for $i = 0, 1, \dots, s-1$ and (ii) p^2 does not divide a_0 .

It is not very difficult to show that every Eisenstein polynomial h in $\mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$. To check irreducibility over $\mathbb{Q}$, Gauss' lemma tells us that it is enough to check irreducibility over $\mathbb{Z}$. So let $f \in \mathbb{Z}[x]$ be irreducible over $\mathbb{Z}$, we say $f(x)$ has an Eisenstein form, if it can be mapped onto an Eisenstein polynomial through a polynomial transformation, i.e. there exists polynomials $\bar{f}, g \in \mathbb{Z}[x]$ such that $f(x) = \bar{f}(x^s)$ for some $s \in \mathbb{Z}_+$ and $\widehat{f(x)} = \bar{f}(g(x))$ is an Eisenstein form of the same degree as f . This transformation is not unique e.g. if $f(x) = x^4 + x^2 + 1$, then $\bar{f}(x) = x^2 + x + 1$ with transformations of the form $x^2 \mapsto x^2 + 3t + 1$, where $t \in \mathbb{Z}$. This yields $\{x^4 + 3(2t + 1)x^2 + 3(3t^2 + 3t + 1) : t \in \mathbb{Z}\}$, a family of 3-Eisenstein forms for f . In this study, we shall preserve the degree, monicity and minimality of f . This restricts us to transformations with smallest coefficients (in terms of absolute values) whose form will be determined by n in $\Phi_n(x)$.

PROPOSITION 1.5. *All order one cyclotomic polynomials have p -Eisenstein forms.*

Proof. (We assume) p is an odd prime. It is enough to check the following 3 cases.

1. Take $n = p^t$ and set $\widehat{\Phi_{p^t}}(x) = \Phi_p((x + 1)^{p^{t-1}})$, then its constant term is $\widehat{\Phi_{p^t}}(0) = \Phi_p(1) = p$ and

$$\widehat{\Phi_{p^t}}(x) = \Phi_p((x + 1)^{p^{t-1}}) = \frac{(x + 1)^{p^t} - 1}{(x + 1)^{p^{t-1}} - 1} \equiv x^{\varphi(p^t)} \pmod{p}.$$

2. Take $n = 2p^t$, Proposition 1.3 part (b); shows $\Phi_{2p^t}(x) = \Phi_{p^t}(-x)$; so $\widehat{\Phi_{2p^t}}(x) = \Phi_p((-x + 1)^{p^{t-1}})$ by case 1, the required Eisenstein form (this is because, in this case, we have $\widehat{\Phi_{2p^t}}(0) = \Phi_p(1) = p$).
3. Take $n = 2^s p^t$, with $s \in \mathbb{Z}_{\geq 2}$, $t \in \mathbb{Z}_+$. By Proposition 1.3, part (a), $\Phi_{2^s p^t}(x) = \Phi_{2p^t}(y)$, where $y = x^{2^{s-1}}$. By case 2 above, $\Phi_{2^s p^t}(x)$ has a p -Eisenstein form and this establishes Proposition 1.5. $\square$

Let $n = 2^s p^t$, the p -Eisenstein form associated to the order one cyclotomic polynomial $\Phi_n(x)$ is

$$\widehat{\Phi_{2^s p^t}}(x) = \begin{cases} \Phi_p((x + 1)^{p^{t-1}}), & s = 0, t \geq 1 \\ \Phi_p((-x^{2^{s-1}} + 1)^{p^{t-1}}), & s, t \geq 1. \end{cases}$$

Here is a list of a few examples of the pairs $(\Phi_n(x), \widehat{\Phi_n}(x))$. Firstly, $\Phi_3(x) = x^2 + x + 1 \mapsto x^2 + 3x + 3$ through $g(x) = x + 1$. Secondly, $\Phi_6(x) = x^2 - x + 1 \mapsto x^2 - 3x + 3$ through $g(x) = -x + 1$ is also a pair. Thirdly, $\Phi_9(x) = x^6 + x^3 + 1 \mapsto x^6 + 6x^5 + 15x^4 + 21x^3 + 18x^2 + 9x + 3$ through $x + 1$. Lastly, we have $\Phi_{36}(x) = x^{12} - x^6 + 1 \mapsto x^{12} - 6x^{10} + 15x^8 - 21x^6 + 18x^4 - 9x^2 + 3$ through $g(x) = -x^2 + 1$, and $\Phi_9(x)$.

From our experiments with SAGE software, it was extremely difficult to find an Eisenstein form for *higher order cyclotomic polynomials*, or perhaps they do not even exist. This motivated the conjecture that *there are infinitely many monic irreducibles in $\mathbb{Z}[x]$ that can not be checked by Eisenstein irreducibility criterion.*

PROPOSITION 1.6. *Let $f \in \mathbb{Z}[x]$ be an order one cyclotomic polynomial with odd prime p , and $\{\widehat{f}(x)\}$ be the set of Eisenstein forms associated to $f(x)$ as described above, then there is a unique transformation (upto modulo p) of the form $\pm x^s + \beta$ that sends $f(x)$ to $\widehat{f}(x)$ i.e. $\widehat{f}(x) = \bar{f}(\pm x^s + \beta)$, where $0 \leq \beta \leq p-1$ and some $s \in \mathbb{Z}_+$.*

Proof. The transformation exists and is a polynomial by Proposition 1.5 (since $\Phi_n(x)$ has integer coefficients over $\mathbb{Q}$). Suppose there are two distinct transformations $x + \beta_1$ and $x + \beta_2$, $0 \leq \beta_1, \beta_2 \leq p-1$ mapping f into two distinct Eisenstein forms, it is clear; all these forms are congruent to $x^{\deg(f)} \pmod{p}$.

1. Consider $f(x) = \widehat{\Phi}_p(x) = \frac{(x+\beta_2)^p-1}{x+\beta_2-1}$, we have $\frac{(x+\beta_2)^p-1}{x+\beta_2-1} \equiv \frac{x^p+\beta_2^p-1}{x+\beta_2-1} \equiv x^{p-1} \pmod{p}$ as $\widehat{\Phi}_p(x)$ is a p -Eisenstein form. Therefore $x^p + \beta_2 - 1 \equiv x^p + (\beta_2 - 1)x^{p-1} \pmod{p}$ implying that $\beta_2 \equiv 1 \pmod{p}$. Therefore, we also have $\beta_1 \equiv 1 \pmod{p}$, hence $\beta_1 = \beta_2$ since both β_1, β_2 belong to $\{0, 1, \dots, p-1\}$.
2. Consider $f(x) = \widehat{\Phi}_{2p}(x) = \frac{((x+\beta_2)^{2p}-1)(x+\beta_2-1)}{(x+\beta_2-1)(x+\beta_2+1)((x+\beta_2)^p-1)}$, we have

$$\begin{aligned} \frac{((x+\beta_2)^{2p}-1)(x+\beta_2-1)}{(x+\beta_2-1)(x+\beta_2+1)((x+\beta_2)^p-1)} &\equiv \\ \frac{((x^p+\beta_2)^2-1)(x+\beta_2-1)}{((x+\beta_2)^2-1)(x^p+\beta_2-1)} &\equiv x^{p-1} \pmod{p}. \end{aligned}$$

Therefore, $((x^p+\beta_2)^2-1)(x+\beta_2-1) \equiv ((x+\beta_2)^2-1)(x^p+\beta_2-1)x^{p-1} \pmod{p}$, implies that $2\beta_2 \equiv (\beta_2-1) \pmod{p}$, $(\beta_2^2-1) \equiv 0 \pmod{p}$ and lastly $(\beta_2^2-1)(\beta_2-1) \equiv 0 \pmod{p}$. All these equations have a common solution only when $\beta_2 \equiv -1 \pmod{p}$. Similarly, $\beta_1 \equiv -1 \pmod{p}$, which means that $\beta_1 = \beta_2$ since both β_1, β_2 belong to $\{0, 1, \dots, p-1\}$.

3. The cases $n = p^t$ and $n = 2^s p^t$ follow from $\Phi_{p^t}(x) = \Phi_p(x^{p^{t-1}})$ and $\Phi_{2^s p^t}(x) = \Phi_{2p}(x^{2^{s-1} p^{t-1}})$. In particular, if $n = 2^s p^t$, $s \neq 0$, then the transformation of the form $-x + \beta$ is used in the proof.

In general, we have a transformation $\pm x^s + \beta$ with β unique upto modulo p . $\square$

The polynomial $\widehat{f}$ is minimal in the sense that, it is obtained from f using the smallest β (in absolute value). We have seen that all order one cyclotomic polynomials have associated p -Eisenstein forms. We now introduce the notion of prime height, in order to characterise the divisibility of the coefficients $\widehat{\Phi}_{2^s p^t}(x)$. Take $n = 2^t p^s$, we define $\mathcal{A}_p(n)$, the p -prime height or valuation of $\Phi_n(x)$, to be the

supremum of the valuation with respect to the prime p of the non-zero coefficients of $\widehat{\Phi}_n(x)$. In other words, $\mathcal{A}_p(n)$ is a ‘measure’ of divisibility of coefficients of $\widehat{\Phi}_n(x)$ with respect to the prime p . We need the following result:

THEOREM 1.7. (Legendre, 1808) *Let p be a prime and let $n = a_0p^t + a_1p^{t-1} + \cdots + a_{t-1}p + a_t$ be the base- p expansion of n . The exact power of a prime p dividing $n!$ (i.e. p -adic valuation) is given by*

$$v_p(n!) = \frac{n - (a_0 + a_1 + \cdots + a_t)}{p - 1}. \quad (3)$$

PROPOSITION 1.8. *If $0 \leq x \leq p^t$, then $v_p\left(\binom{p^t}{x}\right) = t - v_p(x)$.*

Proof. We can deduce facts 1 and 2 below from Theorem 1.7.

1. $v_p(p^t!) = \frac{p^t - 1}{p - 1}$.
2. If $x = \sum_{j=0}^s a_j p^j$, where $0 \leq a_j \leq p - 1$, then $v_p(x!) = \sum_{j=0}^s a_j \frac{p^j - 1}{p - 1}$.
3. If $v_p(x) = i$, then $p^t - x = (p - \sum_{j=i}^s a_j p^{j-i})p^i + (p - 1)p^{i+1} + \cdots + (p - 1)p^{t-1}$, $a_i \neq 0$. So,

$$\begin{aligned} v_p \left(\left(p^t - \sum_{j=i}^s a_j p^j \right)! \right) &= \frac{(p^t - \sum_{j=i}^s a_j p^j) - (-(a_i - 1) - \sum_{j=i+1}^s a_j + (p - 1)(t - i))}{p - 1} \\ &= \frac{p^t - 1}{p - 1} - \sum_{j=i}^s a_j \frac{p^j - 1}{p - 1} - (t - i). \end{aligned}$$

So $v_p\left(\binom{p^t}{x}\right) = \frac{p^t - 1}{p - 1} - \left(\sum_{j=i}^s a_j \frac{p^j - 1}{p - 1}\right) - \left(\frac{p^t - 1}{p - 1} - \sum_{j=i}^s a_j \frac{p^j - 1}{p - 1} - (t - i)\right) = t - i$, as required. $\square$

COROLLARY 1.9. *Let p, t be as above, $0 \leq x \leq p^t a$, where $1 \leq a \leq p - 1$, then $v_p\left(\binom{p^t a}{x}\right) = t - v_p(x)$.*

Proof. Let $v_p(x) = i$, then $v_p(p^t a!) = a \frac{p^t - 1}{p - 1}$ and $v_p((p^t a - x)!) = a \frac{p^t - 1}{p - 1} - \sum_{j=i}^s a_j \frac{p^j - 1}{p - 1} - (t - i)$ and

$$\begin{aligned} v_p \left(\binom{p^t a}{x} \right) &= a \frac{p^t - 1}{p - 1} - \left(\sum_{j=i}^s a_j \frac{p^j - 1}{p - 1} \right) \\ &\quad - \left(a \frac{p^t - 1}{p - 1} - \sum_{j=i}^s a_j \frac{p^j - 1}{p - 1} - (t - i) \right) = t - i. \quad \square \end{aligned}$$

THEOREM 1.10. $\mathcal{A}_p(2^t p^s) = \mathcal{A}_p(p^s) = s$.

Proof. $\widehat{\Phi_{p^s}}(x) = \sum_{i=0}^{p-1} (x+1)^{p^{s-1}i}$. By Proposition 1.8 and Corollary 1.9, it suffices to take the valuation of the coefficient of x in $\widehat{\Phi_{p^s}}(x)$, so $\mathcal{A}_p(p^s) = v_p\left(\sum_{i=0}^{p-1} \binom{p^{s-1}i}{1}\right) = v_p(p^{s-1} \sum_{i=0}^{p-1} i) = s$. For the second part, $\widehat{\Phi_{2^t p^s}}(x) = \Phi_p((-x^{2^{t-1}} + 1)^{p^{s-1}}) = \sum_{i=0}^{p-1} (-x^{2^{t-1}} + 1)^{p^{s-1}i} = \sum_{i=0}^{p-1} (-y + 1)^{p^{s-1}i}$, where $y = x^{2^{t-1}}$. Similarly by Lemma 1.8 and Corollary 1.9, we take the valuation of the coefficient of $x^{2^{t-1}}$ in $\widehat{\Phi_{2^t p^s}}(x)$. A little reflection and relabelling shows that $\mathcal{A}_p(2^t p^s) = v_p\left(\sum_{i=0}^{p-1} \binom{p^{s-1}i}{1}\right) = v_p(p^{s-1} \sum_{i=0}^{p-1} i) = s$. $\square$

The above result shows the worst that can happen is p^s divides at least one of the coefficients.

1.3. Mahler heights. Suppose $f \in \mathbb{C}[z]$, then f factors as $f(z) = \alpha(z - \alpha_1) \cdots (z - \alpha_n)$ over $\mathbb{C}$, (since $\mathbb{C}$ is algebraically closed). The Mahler measure of a polynomial f with respect to the usual absolute value $|\cdot|$ in $\mathbb{C}$ is given by

$$\mathcal{M}(f) = |\alpha| \prod_{i=1}^n \max\{1, |\alpha_i|\}.$$

It is easy to show that Mahler measure as a ‘measure’ is multiplicative, so it makes sense to talk about the Mahler measure of rational functions. For monic polynomials in $\mathbb{C}[z]$, we observe that $\mathcal{M}(f) \geq 1$. A moment’s reflection shows that the following proposition is true.

PROPOSITION 1.11. *For any $f \in \mathbb{C}[z]$, we have $\mathcal{M}(f(z)) = \mathcal{M}(f(-z)) = \mathcal{M}(f(z^s))$.*

The most important result about the Mahler measure of polynomials is due to Kronecker. This asserts,

THEOREM 1.12. (Kronecker’s theorem) *If $f \in \mathbb{Z}[x]$ is monic, $\mathcal{M}(f) = 1$ if and only if f is a product of cyclotomic factors and x^s .*

We now compute Mahler heights of Eisenstein forms of order one cyclotomic polynomials.

LEMMA 1.13. *Let $\gamma(n) = \#\{a : \text{either } 1 \leq a \leq \lfloor \frac{n}{3} \rfloor \text{ or } \lceil \frac{2n}{3} \rceil \leq a \leq n, \text{ and } (a, n) = 1\}$, then*

$$\gamma(n) = \begin{cases} \frac{\varphi(n)}{3}, & \varphi(n) \equiv 0 \pmod{3} \\ \frac{\varphi(n)-1}{3}, & \varphi(n) \equiv 1 \pmod{3} \\ \frac{\varphi(n)+1}{3}, & \varphi(n) \equiv 2 \pmod{3}. \end{cases}$$

Proof. For $n > 2$, the n th-roots of unity over $\mathbb{C}$ occur in complex conjugate pairs, so

$$\begin{aligned}\gamma(n) &= \sum_{d|n} \mu(d) \left\lfloor \frac{\lfloor \frac{n}{3} \rfloor}{d} \right\rfloor = \sum_{d|n} \mu(d) \left(\frac{n}{3d} - \left\{ \frac{n}{3d} \right\} \right) = \frac{\varphi(n)}{3} - \sum_{d|n} \mu(d) \left\{ \frac{n}{3d} \right\} \\ &= \frac{\varphi(n)}{3} - \begin{cases} 0, & \varphi(n) \equiv 0 \pmod{3} \\ \frac{1}{3}, & \varphi(n) \equiv 1 \pmod{3} \\ -\frac{1}{3}, & \varphi(n) \equiv 2 \pmod{3} \end{cases} = \begin{cases} \frac{\varphi(n)}{3}, & \varphi(n) \equiv 0 \pmod{3} \\ \frac{\varphi(n)-1}{3}, & \varphi(n) \equiv 1 \pmod{3} \\ \frac{\varphi(n)+1}{3}, & \varphi(n) \equiv 2 \pmod{3}. \end{cases}\end{aligned}$$

This follows from the bounding $-\frac{1}{3} \leq \sum_{d|n} \mu(d) \left\{ \frac{n}{3d} \right\} \leq \frac{1}{3}$ since $n > 2$ and $\gamma(n) \in \mathbb{Z}_{\geq 1}$. $\square$

As an arithmetic function, $\gamma(n)$ (where $n \geq 3$) counts the number of primitive roots of unity in the first trisection of the unit circle. It is easy to see that $\gamma(n)$ is not a multiplicative function.

THEOREM 1.14. *Let p be an odd prime, $s \in \mathbb{Z}_+$ and $t \in \mathbb{Z}_{\geq 0}$, then*

$$\mathcal{M}(\widehat{\Phi_{2^t p^s}}(x)) = \begin{cases} \prod_{(j, p^s)=1}^{\gamma(p^s)} 4 \sin^2 \left(\frac{\pi j}{p^s} \right), & t = 0 \\ \left(\prod_{(j, 2^t p^s)=1}^{\gamma(2^t p^s)} 4 \sin^2 \left(\frac{\pi j}{2^t p^s} \right) \right)^{\frac{1}{2^t - 1}}, & t \geq 1. \end{cases}$$

Proof. For any odd prime p , the roots of $\widehat{\Phi_p}(x)$ are $\zeta_p^a - 1$, where $a = 1, \dots, p-1$ and ζ_p is the p th root of unity. In this case ($p \geq 3$), the roots of $\Phi_p(x)$ occur in conjugate pairs, therefore to compute the Mahler height of $\widehat{\Phi_p}(x)$, it suffices to consider those cases for which $|\zeta_p^s - 1| \geq 1$. So

$$\begin{aligned}\mathcal{M}(\widehat{\Phi_p}(x)) &= \prod_{j=1}^{p-1} \max\{1, |\zeta^j - 1|\} = \prod_{j=1}^{\gamma(p)} |\zeta^j - 1| |\zeta^{-j} - 1| \\ &= \prod_{j=1}^{\gamma(p)} |2 - (\zeta^j + \zeta^{-j})| = \prod_{j=1}^{\gamma(p)} 4 \sin^2 \left(\frac{\pi j}{p} \right).\end{aligned}$$

Secondly, the Mahler measure formula given sieves out the non-primitive p^s roots of unity by taking on only those j 's that are co-prime to p^s . So

$$\begin{aligned}\mathcal{M}(\widehat{\Phi_{p^s}}(x)) &= \prod_{(j, n)=1}^{\varphi(p^s)} \max\{1, |\zeta^j - 1|\} \\ &= \prod_{(j, n)=1}^{\gamma(p^s)} |2 - (\zeta^j + \zeta^{-j})| = \prod_{(j, n)=1}^{\gamma(p^s)} 4 \sin^2 \left(\frac{\pi j}{p^s} \right).\end{aligned}$$

Lastly, we have $\Phi_{2^t p^s}(x) = \Phi_{2p}(x^{2^{s-1} p^{t-1}}) = \Phi_p(-x^{2^{t-1} p^{s-1}})$ and so $\widehat{\Phi_{2^t p^s}}(x) = \Phi_p((1 - x^{2^{t-1}})^{p^{s-1}})$. In this case, the Mahler measure or height of $\Phi_{2^t p^s}(x)$ is given by (where $1 - \lambda_j^{2^{t-1}} = \zeta^j$)

$$\begin{aligned} \mathcal{M}(\widehat{\Phi_{2^t p^s}}(x)) &= \prod_{(j,n)=1}^{2^t p^s} \max\{1, |\lambda_j|\} \\ &= \prod_{(j,n)=1}^{2^t p^s} \max\{1, |(1 - \zeta^j)|^{\frac{1}{2^{t-1}}}\} = \left(\prod_{(j, 2^t p^s)=1}^{\gamma(2^t p^s)} 4 \sin^2\left(\frac{\pi j}{2^t p^s}\right) \right)^{\frac{1}{2^{t-1}}}. \quad \square \end{aligned}$$

2. Carlitz cyclotomic polynomials.

2.1. The Carlitz module and Carlitz polynomials. Let p be a prime, $q = p^n$ -power of p and $\mathbb{F}_q$ denote finite field of q elements, T an indeterminate. Let $A := \mathbb{F}_q[T]$, $k := \mathbb{F}_q(T)$, $k_\infty := \mathbb{F}_q((\frac{1}{T}))$, k_∞ has a non-archimedean valuation given by $|T|_\infty = q$ and $\overline{k_\infty}$ be the algebraic closure of k_∞ . One may be tempted to think that this is analogous to $\mathbb{C}$ in the sense that it is the algebraic closure of k_∞ (which is taken as the analogue to $\mathbb{R}$). But this is false, because $\overline{k_\infty}$ is neither complete nor locally compact. We resolve the issue of completeness by letting $\mathbb{C}_\infty$ be the completion of $\overline{k_\infty}$, with respect to the canonical extension of $|\cdot|_\infty$ in k_∞ . $\mathbb{C}_\infty$ is now both complete and algebraically closed (but still not locally compact), therefore analogous to $\mathbb{C}$ (in the sense of doing arithmetic).

Let τ be the Frobenius element defined by $\tau(x) = x^q$. We denote by $k\{\tau\}$ the ‘twisted polynomial ring’ whose multiplication is defined by $\tau\alpha = \alpha^q\tau$ for all $\alpha \in k$. The Carlitz A -module is the abelian group $\mathbb{C}_\infty$ together with the $\mathbb{F}_q$ -algebra homomorphism $\rho : A \rightarrow k\{\tau\}$ defined by; for each $m \in A$, $m \mapsto \rho_m$, and is characterised by $\rho_T = \tau + T$. To each $m \in A$ and $x \in \mathbb{C}_\infty$, we set $m * x := \rho_m(x)$ giving $\mathbb{C}_\infty$ a new A -module structure denoted by $\mathcal{C} := \rho(\mathbb{C}_\infty)$. Like in the classical case, to each $m \in A_+$, we associate an additive and separable polynomial $\rho_m(x)$; the m -Carlitz polynomial. This is analogous to $x^n - 1$ in the classical setting.

Let $m \in A$, then $\mathbb{C}_\infty[m] := \{\lambda \in \mathbb{C}_\infty : \rho_m(\lambda) = 0\}$ is the set of Carlitz m -torsion points. As A -modules, $\mathbb{C}_\infty[m]$ is isomorphic to A/mA through the Carlitz module ρ . Now, an element $\lambda \in \mathbb{C}_\infty[m]$ is called a primitive torsion point of m if and only if it generates $\mathbb{C}_\infty[m]$ as an A -module. Adjoining $\mathbb{C}_\infty[m]$ to k produces a Galois extension $K_m := k(\mathbb{C}_\infty[m])$, called the *m -Carlitz-cyclotomic function field*. This is to differentiate it from the classical cyclotomic extensions obtained by adjoining roots of unity to $\mathbb{Q}$.

DEFINITION 2.1. Let $0 \neq m \in A$, the Carlitz m -cyclotomic polynomial over k the polynomial,

$$\Phi_m(x) := \prod_{\lambda: \text{primitive } m \text{ torsion}} (x - \lambda). \quad (4)$$

The emphasis on the name m -Carlitz cyclotomic polynomial to distinguish it from the elementary n th-cyclotomic polynomial $\Phi_n(x)$. Like $\Phi_n(x)$, $\Phi_m(x)$ satisfies nice relations which motivates the next section:

2.2. Some properties of Carlitz cyclotomic polynomials. We begin with a recursive formula for computing the coefficients of $\rho_m(x)$.

LEMMA 2.2. ([1], Proposition 3.3.10) For any $m \in A$, $\rho_m(x) = a_0x + a_1x^q + \cdots + a_nx^{q^{\deg(m)}}$, where

$$a_0 = m, a_i = \frac{a_{i-1}^q - a_{i-1}}{T^{q^i} - T}, \quad i = 1, \dots, \deg(m).$$

It is easy to check that, if $m = P^s$, then $\rho_m(\tau)$ is Eisenstein at the prime P , atleast as a polynomial in τ . Like in Theorem 1.7, as seen before, the most important ingredient in the proof of the rational function field analogue for Theorem 1.10 is Lemma 2.2. However, this is not the A -analogue of Legendre's Theorem 1.7.

PROPOSITION 2.3.

$$\rho_m(x) = \prod_{D|m} \Phi_D(x)$$

where the product runs over all monic divisors of m .

Proof. Imitate the proof of an analogous statement for elementary cyclotomic polynomials [5]. $\square$

Applying the (multiplicative form of the) Möbius inversion formula, we obtain

$$\Phi_m(x) = \prod_{D|m} (\rho_{\frac{m}{D}}(x))^{\mu(D)}. \quad (5)$$

where μ is the polynomial Möbius function defined as $\mu(m) = (-1)^s$ if m is square free with s distinct primes, otherwise $\mu(m) = 0$. The identity in equation (5) helps us to study properties of $\Phi_m(x)$. In fact one can show that these cyclotomic polynomials obtained in this way are also irreducible over k , and have integral coefficients, and degree $\varphi(m)$, where φ is the analogue of the Euler totient function. We will later show that $\Phi_P(x) \in A[x]$ is indeed a P -Eisenstein polynomial, atleast as a polynomial in x^{q-1} . Therefore, $\Phi_P(x)$ is analogous to $\widehat{\Phi}_p(x)$ and $0 \neq \lambda_P \in \mathbb{C}_\infty[P]$ is analogous to $\zeta_p - 1$ and not ζ_p as one would expect. In fact this observation is very important and is even more pronounced when one studies ramification of prime ideals in Carlitz cyclotomic function fields. From now on, $s \in \mathbb{Z}_+$, $m \in A_+$ and P a prime in A .

PROPOSITION 2.4.

$$\Phi_{mP^s}(x) = \begin{cases} \Phi_m(\rho_{P^s}(x)), & (m, P) \neq 1 \\ \Phi_{mP}(\rho_{P^{s-1}}(x)), & (m, P) = 1. \end{cases}$$

Proof. Suppose $(m, P) \neq 1$, this means $P \mid m$.

$$\begin{aligned}\Phi_{mP^s}(x) &= \prod_{D \mid mP^s} \left(\rho_{\frac{mP^s}{D}}(x) \right)^{\mu(D)} \\ &= \Phi_m(\rho_{P^s}(x)) \prod_{D \mid mP^s, D \nmid m} \left(\rho_{\frac{mP^s}{D}}(x) \right)^{\mu(D)} = \Phi_m(\rho_{P^s}(x))\end{aligned}$$

since $D \mid mP^s$ and $D \nmid m$ implies $P^2 \mid D$, therefore $\mu(D) = 0$. Now suppose $P \nmid m$,

$$\begin{aligned}\Phi_{mP^s}(x) &= \prod_{D \mid mP^s} \left(\rho_{\frac{mP^s}{D}}(x) \right)^{\mu(D)} \\ &= \Phi_{mP}(\rho_{P^{s-1}}(x)) \prod_{D \mid mP^s, D \nmid mP} \left(\rho_{\frac{mP^s}{D}}(x) \right)^{\mu(D)} = \Phi_{mP}(\rho_{P^{s-1}}(x)),\end{aligned}$$

again $D \mid mP^s$ and $D \nmid mP$ implies $P^2 \mid D$, therefore $\mu(D) = 0$ and the result follows. $\square$

PROPOSITION 2.5.

$$\Phi_m(0) = \begin{cases} 1, & \text{if } m \text{ is not a power of a prime,} \\ P, & m = P^s. \end{cases}$$

Proof. Imitate the proof of an analogous statement for elementary cyclotomic polynomials [5]. $\square$

Although the properties are almost the same, it is important to point out that reciprocity does not always hold and so palindromy is one of the properties that have no direct analogues over rational function field.

2.3. Coefficients of Carlitz cyclotomic polynomials. Let $m \in A$, the order of $\Phi_m(x)$ is the number of distinct prime factors of m . If m has only one prime factor P , then the P -prime height of $\Phi_m(x)$, $\mathcal{A}_P(m)$ as the maximum valuation with respect to P of the non-zero coefficients of $\Phi_m(x)$. In this setting, the P -prime height of $\Phi_m(x)$ is defined analogous to the classical case because the order one Carlitz cyclotomic polynomials are already Eisenstein. This can be deduced by induction starting with Lemma 2.2 and Proposition 2.4. For a rigorous proof see [3], Corollary 12.7. This property of order one Carlitz cyclotomic polynomials helps us get the analogy close with the Eisenstein forms discussed in Section 1. There seems to be no clever transformation to create other Eisenstein forms.

THEOREM 2.6. For any prime $P \in A$, we have $\mathcal{A}_P(P) = 1$.

Proof. Suppose $\deg(P) = 1$, then $\Phi_P(x) = x^{q-1} + P$, clearly its valuation set is $V_P = \{0, 1\}$, (we only take the valuation of the non zero terms) therefore $\mathcal{A}_P(P) = 1$. Suppose $\deg(P) = n > 1$, then by Proposition 2.2, $\Phi_P(x) = a_0 + a_1x^{q-1} + \dots + a_nx^{q^n-1}$, where the coefficients are given by

$$a_0 = P, a_1 = \frac{a_0^q - a_0}{T^q - T}, a_2 = \frac{a_1^q - a_1}{T^{q^2} - T}, \dots, a_{n-1} = \frac{a_{n-2}^q - a_{n-2}}{T^{q^{n-1}} - T}, a_n = 1.$$

Observe that $v_P(a_0) = 1$ and $v_P(a_n) = 0$. Now, $v_P(a_0^q - a_0) = v_P(P^q - P) = 1$ since P does not divide $(P^{q-1} - 1)$ and $v_P(T^q - T) = 0$ since $\deg(P) > 1$ and $T^q - T$ splits over $\mathbb{F}_q$, so $v_P(a_1) = 1$. Similarly, we have $v_P(a_2) = v_P(a_1^q - a_1) - v_P(T^{q^2} - T) = 1$. This is done until a_{n-1} is reached since a prime P divides $T^{q^n} - T$ if and only its degree divides n . $v_P(a_n) = v_P(a_{n-1}^q - a_{n-1}) - v_P(T^{q^n} - T) = 0$, because at this point, at-least one of the factors of $T^{q^n} - T$ is P . Therefore, the valuation set of $\Phi_P(x)$ is $V_P = \{0, 1\}$. $\square$

We now attempt to find the prime height for $m = \alpha P^s$, where $\alpha \in \mathbb{F}_q^*$ and set $\Phi_{\alpha m}(x) := \alpha \Phi_m(x)$.

THEOREM 2.7. *Let $P \in A_+$ be a prime of degree n , $s \in \mathbb{Z}_{\geq 2}$, then $\mathcal{A}_P(P^s) = (q^n - 1)(s - 1)$.*

Proof. We proceed by induction on powers s of P . Trivially, $\Phi_1(x) = x$ and $\Phi_P(x) = x^{q^n-1} + a_1x^{q^{n-1}-1} + \dots + a_{q(q^{n-1}-1)}x^{q-1} + P$, where by Theorem 2.6, we have $\mathcal{A}_P(P) = 1$. In particular $v_P(a_{q(q^{n-1}-1)}) = 1$. Since both $\rho_P(x)$ and $\Phi_P(x)$ have the same non-zero coefficients (all divisible by P but not P^2), and $q^n - 1 > q^{n-1}$ for all $q, n \in \mathbb{Z}_+$. Then $(\rho_P(x))^{q^n-1}$ contains the term with the highest valuation w.r.t. P below; so

$$\Phi_{P^2}(x) = \Phi_P(\rho_P(x)) = (\rho_P(x))^{q^n-1} + \dots + P = x^{q^n-1} \left(\Phi_P(x)^{q^n-1} \right) + \dots + P.$$

The coefficient of x^{q^n-1} in $\Phi_{P^2}(x)$ is given by $\Phi_P(0)^{q^n-1} = P^{q^n-1}$. Therefore, when $s = 2$, we have $\mathcal{A}_P(P^2) = q^n - 1 = (q^n - 1)(2 - 1)$, and so the formula true for $s = 2$. Suppose it is true for $s = n'$, that is to say $\mathcal{A}_P(P^{n'}) = (q^n - 1)(n' - 1)$ and the coefficient attaining this valuation is x^{q^n-1} . We now compute $\Phi_{P^{n'+1}}(x)$. Applying Theorem 2.4, we obtain

$$\begin{aligned} \Phi_{P^{n'+1}}(x) &= \Phi_P(\rho_{P^{n'}}(x)) = \rho_{P^{n'}}(x)^{q^n-1} + \dots \\ &\dots + P = x^{q^n-1} \left(\prod_{t=1}^{n'} \Phi_{P^t}(x)^{q^{n-1}} \right) + \dots + P, \end{aligned}$$

with the position having maximum valuation (with respect to P) at x^{q^n-1} . To obtain the coefficient of x^{q^n-1} , consider the constant terms of $\Phi_{P^t}(x)$. So the coefficient of x^{q^n-1} in $\Phi_{P^{n'+1}}(x)$ is given by the expression $\prod_{t=1}^{n'} \Phi_{P^t}(0)^{q^{n-1}} = P^{n'(q^n-1)}$, so $\mathcal{A}_P(P^{n'+1}) = (q^n - 1)(n' + 1 - 1)$ which completes the proof. $\square$

Theorem 2.6 and Theorem 2.7 give a complete analogue to Theorem 1.10.

EXAMPLE 2.8. Over $\mathbf{F}_3[T]$, with $P = T$. Computations using **SAGE** yield, $\Phi_{T^4}(x) = x^{54} + (2T^9 + 2T^3 + 2T)x^{36} + (2T^6 + 2T^4 + 2T^2)x^{30} + 2T^3x^{28} + (T^{18} + 2T^{12} + 2T^{10} + T^6 + 2T^4 + T^2)x^{18} + (2T^{15} + 2T^{13} + 2T^{11} + 2T^9 + T^7 + T^5 + 2T^3)x^{12} + (2T^{12} + 2T^6 + 2T^4)x^{10} + (T^{12} + 2T^{10} + 2T^6 + T^4)x^6 + (2T^9 + 2T^7 + 2T^5)x^4 + T^6x^2 + T$. The coefficient with highest valuation with respect to the prime T is T^6 and so $\mathcal{A}(P^4) = 6 = (3^1 - 1)(4 - 1)$.

2.4. Logarithmic and Mahler heights of $\Phi_m(x)$ and $\rho_m(x)$. We shall work with $|\cdot|_\infty$, the absolute value corresponding to the place at infinity. For every $m \in A$, we have $|m|_\infty = q^{\deg(m)}$. In this case, a polynomial has a large size if it has high degree and vice versa.

DEFINITION 2.9. Let $f(x) = \sum_{i=0}^n c_i x^i$, the logarithmic height of f is $\mathcal{H}(f) = \log_q(\max_{i \leq n} \{|c_i|_\infty\})$.

We have already seen that order of $\Phi_m(x)$ depends on the underlying field. This is because, an element may be irreducible over one ring but reducible over the other e.g. $f(T) = T^2 + 1$ is irreducible in $\mathbb{F}_3[T]$ but not in $\mathbb{F}_5[T]$. So $\Phi_{T^3+T}(x)$ is of order 2 in $\mathbb{F}_3[T]$ and order 3 over $\mathbb{F}_5[T]$. In this case, $f(x) \in A(x)$ is said to be flat if $\mathcal{H}(f) = 0$ i.e. all its coefficients are units in A . For-example, in $\mathbb{F}_q[T]$ with $q \geq 3$, $f(x) = x^{q^2} + 2x^q + 1$ is flat whereas $g(x) = x^{q^2} + (T^q + T)x^q + 1$ is non flat. Now, since the absolute values in A are non-archimedean, the results differ from one ring of integers to another. Unlike the classical case where $\mathcal{H}(\Phi_n(x)) = \mathcal{H}(\Phi_{N_0}(x))$, with N_0 being the odd square-free part of n . Over A , this is false.

PROPOSITION 2.10. ([2], p. 140) *Let F be a non archimedean valued field. For all $0 \neq f, g \in F[x]$, $\mathcal{H}(fg) = \mathcal{H}(f) + \mathcal{H}(g)$.*

Proof. Follows from multiplicativity of the height function in non-archimedean absolute valuation [2]. $\square$

From Proposition 2.10, we then have the following propositions and theorems,

PROPOSITION 2.11. *Let $m \in A$ with $\deg(m) \geq 1$, then $\mathcal{H}(\rho_m(x)) = q^{\deg(m)-1}$.*

Proof. Let $m \in A$ with $\deg(m) = n$, then $\rho_m(x) = \sum_{j=1}^n c_j x^{q^j}$ with $\deg(c_j) = q^j(n-j)$. If $f \in A$ then $|f|_\infty = q^{\deg(f)}$, so $|c_j|_\infty = q^{q^j(n-j)}$. Consider the real valued function $y = q^x(n-x)$ where $q, n \in \mathbb{N}$ are fixed (with $q \geq 2$). In this case, the maximum value is obtained at $x = n - \frac{1}{\ln(q)}$. Since we require integer values, we have two possible values either at $x = n$ or $n-1$, when $q \geq 3$. In this case $y = 0$ or $y = q^n$, and clearly the maximum is obtained when $x = n-1$. When $q = 2$, we have $x = n-1$ or $n-2$; in both cases we obtain $y = 2^{n-1}$ hence, we either take $j = n-1$ or $n-2$. Hence $\mathcal{H}(\rho_m(x)) = \log_q(|c_{n-1}|_\infty) = q^{n-1}$. $\square$

COROLLARY 2.12. $\mathcal{H}(\rho_1(x)) = \mathcal{H}(\Phi_1(x)) = 0$ and $\mathcal{H}(\rho_P(x)) = \mathcal{H}(\Phi_P(x)) = q^{\deg(P)-1}$.

THEOREM 2.13. $\mathcal{H}(\Phi_m(x)) = \sum_{D|m} q^{\deg(D)-1} \mu\left(\frac{m}{D}\right)$.

Proof.

$$\begin{aligned} \mathcal{H}(\Phi_m(x)) &= \sum_{D|m} \mathcal{H}(\rho_D(x)) \mu\left(\frac{m}{D}\right) \\ &= \sum_{D|m} q^{\deg(D)-1} \mu\left(\frac{m}{D}\right) = \sum_{D|m} q^{\deg(D)-1} \mu\left(\frac{m}{D}\right). \end{aligned} \quad \square$$

For-example, over $\mathbb{F}_3[T]$, take $P_1 = T^2 + 1$, $P_2 = T^2 + 2T + 2$ for our primes, and set $m = P_1^2 P_2$. Our brute force algorithm using **SAGE** returns the logarithmic height of $\Phi_m(x)$ as 192. Writing this polynomial explicitly would require another 15 more pages of the manuscript. With the above formula, we can compute this value quickly without worrying about the involved “monster polynomials”. We have

$$\begin{aligned} \mathcal{H}(\Phi_m(x)) &= 3^{2(2)+2(1)-1} + 3^{2(2-1)+2(1-1)-1} - 3^{2(2-1)+2(1)-1} - 3^{2(2)+2(1-1)-1} \\ &= 192. \end{aligned}$$

This shows that we can’t classify Carlitz cyclotomic polynomials according to height based on order (as in the classical case). In fact, the classical result of flatness for order 1 and 2 cyclotomic polynomials is false.

Lastly, we discuss Mahler heights of $\rho_m(x)$. Suppose $f \in \mathbb{C}_\infty[z]$, since $\mathbb{C}_\infty$ is algebraically closed, f factors as $f(z) = \alpha(z - \alpha_1) \cdots (z - \alpha_n)$ over $\mathbb{C}_\infty$ where $\alpha \in \mathbb{C}_\infty^*$. We define the Mahler measure of $f(z)$ as

$$\mathcal{M}(f) := \mathcal{M}(f(z)) = |\alpha|_\infty \prod_{i=1}^n \max\{1, |\alpha_i|_\infty\}.$$

Through-out this section, we take f to be monic and therefore $\mathcal{M}(f) = \prod_{i=1}^n \max\{1, |\alpha_i|_\infty\}$. Since Mahler measure is multiplicative, Proposition 2.3 shows that to determine $\mathcal{M}(\rho_m(x))$, it suffices to find Mahler measure of its Carlitz cyclotomic factors $\Phi_a(x)$. Since all the roots of a given Carlitz cyclotomic polynomial are conjugate, they must have the same norm and therefore absolute value. So, in principle, we only need to find the norm of any generator (or any primitive m -torsion point) of $\mathbb{C}_\infty[m]$ as an A -module.

PROPOSITION 2.14. ([3], Proposition 12.6) *Let λ_m be a $\mathbb{C}_\infty[m]$ -generator and suppose, $a \in A$ is co-prime to m . Then $\frac{\rho_a(\lambda_m)}{\lambda_m}$ is a unit in $\mathcal{O}_m$, the ring of integers of $K_m := k[\lambda_m]$. Moreover, if m is of order ≥ 2 , then λ_m is itself a unit.*

PROPOSITION 2.15. *Let $m \in A$ be of order ≥ 2 , then $\mathcal{M}(\Phi_m(x)) = 1$.*

Proof. Proposition 2.14 asserts that, if $m \in A$ is composite, then the generators of $\mathbb{C}_\infty[m]$ are units in $\mathcal{O}_m$, therefore all the conjugates are units and so have absolute value 1. Hence $\mathcal{M}(\Phi_m(x)) = 1$. $\square$

This is almost analogous to the classical result, whereby the Mahler height of $g(x) \in \mathbb{C}[x]$ is 1 if and only if atleast one of the roots of $g(x)$ lies on (others may lie inside) the unit circle (as demonstrated in Kronecker's theorem). For order 1 cyclotomic polynomials, we have the following result:

PROPOSITION 2.16. *Let $P \in A$ be a prime polynomial, then $\mathcal{M}(\Phi_P(x)) = q^{\deg(P)}$.*

Proof. Proposition 2.14 asserts that for every prime $P \in A$, with λ as a generator of the Carlitz P -torsion points, we have $\frac{\rho_b(\lambda)}{\lambda} \in \mathcal{O}_P^*$ for all $b \in A$ co-prime to P . Since all these P -torsions are non-zero integral functions, (non units), their norms in k must be different from 0 and 1 i.e. absolute values are > 1 . Thus,

$$\mathcal{M}(\Phi_P(x)) = \prod_{i=1}^{\varphi(P)} \max\{1, |\lambda_i|\} = \left| P^{\frac{1}{[K_P:k]}} \right|^{\varphi(P)} = \left| P^{\frac{1}{\varphi(P)}} \right|^{\varphi(P)} = q^{\deg(P)}. \quad \square$$

COROLLARY 2.17. *Let $s \in \mathbb{Z}_+$, $m \in A$, and P be a prime, then $\mathcal{M}(\Phi_{P^s}(x)) = q^{\deg(P)}$ and $\mathcal{M}(\rho_m(x)) = q^{\deg(m)}$.*

Proof. Since $\Phi_{P^s}(0) = P$ by Proposition 2.5, $\mathcal{M}(\Phi_{P^s}(x)) = q^{\deg(P)}$. Secondly, $\mathcal{M}(\rho_m(x)) = q^{\deg(m)}$ follows from using Proposition 2.15 together with the fact that Mahler height function $\mathcal{M}$ multiplicative. $\square$

Combining Propositions 2.15, 2.16 with Corollary 2.17, we obtain Theorem 2.18.

COROLLARY 2.18. (Partial Kronecker's theorem) *Let $g(x) \in A[x]$ be a non constant polynomial, $\alpha \in \mathbb{F}_q^*$, then if $g(x)$ is a product of powers of $x - \alpha$ and Carlitz cyclotomic polynomials, then $\mathcal{M}(g(x)) = q^n$ for some $n \in \mathbb{Z}_+$.*

This is analogous to the forward part of the classical theorem due to Kronecker, if this is not true, then we have to rethink about the right arithmetic analogue of Kronecker theorems over the rational function field k . The converse to Theorem 2.18 is in general false. For example, $g(x) = x - T^n$ has Mahler measure q^n but it is neither cyclotomic nor a product of cyclotomic polynomials and x .

Conclusion. Utilising the number fields-function fields analogy, we defined the order and P -height of Carlitz cyclotomic polynomials. Motivated by classical results on the ' p -prime height' for order 1 cyclotomic polynomials, we showed that over $\mathbb{Q}$, $\mathcal{A}_p(p^s) \propto s$, where as over k , $\mathcal{A}_P(P^s) \propto (q^{\deg(P)} - 1)(s - 1)$. We investigated logarithmic and Mahler heights in both cases and gave a weak form of Kronecker's theorem.

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