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ON THE LOCATION OF RATIOS OF ZEROS OF SPECIAL TRINOMIALS¹

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ABSTRACT. Given coprime integers k, ℓ with $k > \ell \geq 1$, and arbitrary complex polynomials $A(z)$ and $B(z)$ such that $\deg(A(z)B(z)) \geq 1$, we consider the polynomial sequence $\{P_n(z)\}$ satisfying the three-term recurrence relation

$$P_n(z) + B(z)P_{n-\ell}(z) + A(z)P_{n-k}(z) = 0,$$

subject to the initial conditions $P_0(z) = 1$, $P_{-1}(z) = \cdots = P_{1-k}(z) = 0$. We prove the inequality condition for the real algebraic curve on which the zeros of the polynomials in the sequence $\{P_n(z)\}$ lie. Furthermore, we demonstrate that for any $n \in \mathbb{Z}_{\geq 1}$ and a zero z_0 of $P_n(z)$ with $A(z_0) \neq 0$, at least two of the distinct zeros of the trinomial

$$D(t; z_0) := A(z_0)t^k + B(z_0)t^\ell + 1$$

have a ratio that either lies on the real line or on the unit circle centered at the origin. This reveals a geometric property exhibited by the zeros of trinomials of the form $t^k + at^\ell + 1$, where $a \in \mathbb{C} - \{0\}$ and $a^k \in \mathbb{R}$.

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Key words: Trinomial, null-collinearity, q -discriminant, real-rootedness, reciprocal polynomials.

1. Basic notions and main results. Recursive polynomial sequences have long been a subject of interest due to their easy definition, computation and extensive applications in various fields, such as mathematics, physics, computer science, engineering and statistics, [4]. The recursive structure of these polynomials allows them to model a wide variety of phenomena as explained below. For example,

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in physics, recursive polynomials often arise as solutions to specific ordinary differential equations. In particular, the Hermite polynomials $H_n(x)$ (satisfying the linear recurrence relation $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ for $n \in \mathbb{Z}_{\geq 1}$ subject to $H_0(x) = 1$ and $H_1(x) = 2x$) are pivotal in quantum mechanics and statistical mechanics, where they describe the solutions to the Schrödinger equation for a one dimensional harmonic oscillator, see [4] for details.

In combinatorics, recursive polynomials are closely related to various combinatorial structures, like permutations, graphs, and partitions. For example, Eulerian polynomials satisfy the recurrence $A_n(x) = (1-x)A_{n-1}(x) + x(1-x)^{n-1}$, with the initial condition $A_0(x) = 1$. The n -th Eulerian polynomial is given by

$$A_n(x) := \sum_{k=0}^n A(n, k)x^k, \text{ where } A(n, k) \text{ counts the number of permutations of } n$$

objects with exactly k ascents, [14]. Other recursive polynomials include umbral polynomials such as rising and falling factorials, the Bernoulli, Dickson and Touchard polynomials, e.t.c.

Some of the above-mentioned polynomial sequences like the Hermite are examples of orthogonal polynomial sequences that satisfy three-term recurrence relations. Three-term recurrence relations are fundamental in the study of orthogonal polynomials, as they provide an efficient method for generating orthogonal polynomial sequences with respect to a suitable weight function, see [5] and [17] for details. In the same papers, it is shown that orthogonality is a desirable property in representations of functionals used in efficient numerical computations. Furthermore, the orthogonality of these polynomials is used to prove other properties of the polynomial sequence, for example real-rootedness, see [5] and [8]. Nonetheless, it is important to note that there do exist other orthogonal polynomial sequences which arise from higher order recurrences, for example the generalised Jacobi polynomials, generalised Hermite polynomials, generalised Laguerre polynomials and Askey-Wilson polynomials among others. These are less common but can appear in specific settings, such as generalized orthogonal polynomials, multivariate polynomials, and quantum group theory, see [6] for details.

It is almost a tradition in mathematics that when studying a family of polynomials, particularly those generated by recursions, one is interested in exploring properties such as orthogonality, equimodularity, unimodularity, real-rootedness, stability, and interlacing of roots. These properties have important implications in various branches of pure and applied mathematics, as detailed in [3]. Several authors (for example, [2, 9, 10, 19, 20, 21]) have focused on the problem of determining the location of the zeros of polynomials within polynomial sequences, especially those generated by linear recurrence relations. This interest arises from the fact that understanding the zeros of certain polynomials in a sequence can provide valuable information about the zeros of other polynomials within the same sequence. In particular, these studies aim to either establish criteria for the real-rootedness of polynomial sequences or to explicitly determine the limiting curve on which the zeros of the polynomials lie, see [2, 10, 19, 20] for details.

The location of the zeros of a sequence of polynomials is vital for various scientific and engineering applications. It aids in solving polynomial equations, factoring

polynomials, and understanding their behavior. Knowledge of the zeros improves the accuracy of numerical methods, such as polynomial interpolation, and enhances the efficiency of root-finding algorithms. Understanding polynomial zeros helps prevent numerical instability and ensures reliable computational methods. In control theory, the zeros are crucial for stability analysis, while in optimization, they help identify optimal points [12]. In signal processing, the knowledge of location of zeros is essential for filter design, [13].

The goal of this paper is to extend the knowledge on the location of zeros of recursive polynomials. In this framework, we let k and ℓ be coprime integers with $k > \ell \geq 1$, $A(z)$ and $B(z)$ be arbitrary non-zero complex polynomials such that $\deg(A(z)B(z)) \geq 1$. We then consider the polynomial sequence $\{P_n(z)\}$ satisfying the following three-term recurrence

$$P_n(z) + B(z)P_{n-\ell}(z) + A(z)P_{n-k}(z) = 0, \quad (1)$$

subject to the initial conditions $P_0(z) = 1$ and $P_{-1}(z) = \cdots = P_{1-k}(z) = 0$. If we take a solution of the form $P_n(z) = (t(z))^n$ where $n \in \mathbb{Z}_{\geq 0}$ and $t(z)$ is a nonzero rational function, then substituting it in recurrence (1) yields

$$\begin{aligned} 0 &= (t(z))^n + B(z)(t(z))^{n-\ell} + A(z)(t(z))^{n-k} \\ &= ((t(z))^k + B(z)(t(z))^{k-\ell} + A(z))(t(z))^{n-k}. \end{aligned}$$

Since $t(z) \neq 0$, it follows that, $(t(z))^k + B(z)(t(z))^{k-\ell} + A(z) = 0$ except probably for finitely many $z \in \mathbb{C}$. The equation $t^k + B(z)t^{k-\ell} + A(z) = 0$ (with $t(z)$ written as t) is called the characteristic equation of recurrence relation in (1), while $\Delta(t; z) := t^k + B(z)t^{k-\ell} + A(z)$ is its characteristic polynomial. In [10, Lemma 1], the ordinary generating function of the polynomial sequence $\{P_n(z)\}$ generated by the recurrence relation (1) is shown to be

$$G(t; z) := \sum_{n=0}^{\infty} P_n(z)t^n = \frac{1}{1 + B(z)t^\ell + A(z)t^k}.$$

We shall denote the denominator of $G(t; z)$ by $D(t; z) := A(z)t^k + B(z)t^\ell + 1$, the reciprocal polynomial of $\Delta(t; z)$.

REMARK 1.1. We consider k and ℓ to be coprime since the case for non-coprime integers, does not provide any new information about the distribution of the zeros of non-zero polynomials in its sequence. To see this, we take $r, s \in \mathbb{Z}_{\geq 1}$ with $r = \ell d$, $s = kd$, $\gcd(k, \ell) = 1$ and $k > \ell \geq 1$. The ordinary generating function $G_R(t; z)$ for the polynomial sequence $\{R_m(z)\}$ generated by the three-term recurrence $R_n(z) + B(z)R_{n-r}(z) + A(z)R_{n-s}(z) = 0$, subject to the initial conditions $R_0(z) = 1$, $R_{-1}(z) = \cdots = R_{1-s}(z) = 0$ is related to $G(t; z)$ as follows:

$$\begin{aligned} G_R(t; z) &= \sum_{m=0}^{\infty} R_m(z)t^m = \frac{1}{1 + B(z)t^r + A(z)t^s} = \frac{1}{1 + B(z)w^{\ell d} + A(z)w^{kd}} \\ &= \sum_{n=0}^{\infty} P_n(z)t^{nd} = G(t^d; z). \end{aligned}$$

Therefore, $R_m(z) = P_n(z)$ if $m = nd$ and $R_m(z) = 0$ if $d \nmid m$, i.e., the sequence $\{R_m(z)\}$ is “essentially” $\{P_n(z)\}$.

Of all the results in the literature regarding the location of (the algebraic curve containing) the zeros of $\{P_n(z)\}$, the most general and crucial to us is [2, Theorem 1.1] due to Bögvad, et. al. To state it, we need the following notation: For a non-empty subset $\mathcal{X}$ of nonzero complex polynomials, we set $\mathcal{Z}_{\mathcal{X}} := \{z^* \in \mathbb{C} : f(z^*) = 0 \text{ for some } f \in \mathcal{X}\} = \bigcup_{f(z) \in \mathcal{X}} \mathcal{Z}_{f(z)}$ (or $\mathcal{Z}_{A(z)}$ if $\mathcal{X} = \{A(z)\}$).

In addition, we define the following curves parametrised by k and ℓ : $\Gamma_{k,\ell} := \left\{z \in \mathbb{C} : \operatorname{Im} \left((-1)^k \frac{B^k(z)}{A^\ell(z)} \right) = 0 \right\}$, $\Gamma_{k,\ell}^* := \left\{z \in \mathbb{C} : 0 \leq \operatorname{Re} \left((-1)^k \frac{B^k(z)}{A^\ell(z)} \right) < \frac{k^k}{(k-\ell)^{k-\ell}} \right\}$ and $\mathcal{H}_{k,\ell} := \Gamma_{k,\ell} \cap \Gamma_{k,\ell}^*$.

THEOREM 1.2. ([2, Theorem 1.1]) *If $\{P_n(z)\}$ is the polynomial sequence generated by recurrence relation (1) and $z_0 \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$, then z_0 lies on the real algebraic curve $\Gamma_{k,\ell}$.*

REMARK 1.3. We have the following remarks.

1. Theorem 1.2 generalizes the specific cases $k = 2, 3$ and 4 with $\ell = 1$, see [19, Theorems 1, 3, 5] respectively.
2. As n tends to ∞ , the zeros of $P_n(z)$ become dense in $\mathcal{H}_{k,\ell}$ for $(k, \ell) \in \{(2, 1), (3, 1), (4, 1)\}$, see [19, Theorems 1, 3, 5]. For $k \geq 5$ and $\ell = 1$, the zeros of $P_n(z)$ become dense in $\mathcal{H}_{k,\ell}$ for sufficiently large n , see [20, Theorem 1].
3. The inequality condition in $\Gamma_{k,\ell}^*$ for the case $k \geq 5$ and $\ell > 1$ is of a different flavor as stated in Theorem 1.4. We will provide its proof in Subsection 2.2 of this paper.

Theorem 1.2 gives a partial characterisation of the real algebraic curve on which the zeros of the generated polynomials lie. This motivated the authors to search for the full inequality condition of the curve $\Gamma_{k,\ell}$ as stated in Theorem 1.4. However, the authors acknowledge that, the density condition is still out of their reach by current approaches.

THEOREM 1.4. *If $\{P_n(z)\}$ is the polynomial sequence generated by (1) with $\ell > 1$ and $z_0 \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$, then z_0 lies on the curve*

$$\Gamma_1 := \left\{ \begin{cases} \left\{ z \in \Gamma_{k,\ell} : \operatorname{Re} \left((-1)^{k-\ell} \frac{B^k(z)}{A^\ell(z)} \right) \leq 0 \right\}, & \text{where } \mathcal{Z}_{\{B(z)\}} - \mathcal{Z}_{A(z)} \neq \emptyset, \\ \left\{ z \in \Gamma_{k,\ell} : \operatorname{Re} \left((-1)^{k-\ell} \frac{B^k(z)}{A^\ell(z)} \right) < 0 \right\}, & \text{where } \mathcal{Z}_{\{B(z)\}} \subset \mathcal{Z}_{A(z)}. \end{cases} \right.$$

Another problem that we deal with in the present paper concerns the location of zeros of trinomials. In general, the problem of the number and location of zeros of an arbitrary trinomial $at^k + bt^\ell + c$ (where $a, b, c \in \mathbb{C} - \{0\}$ and $k, \ell \in \mathbb{Z}$ such that

$k \geq 3$, $k > \ell \geq 1$ with k and ℓ coprime) has a long history of study starting with J. Lambert in 1758, followed by L. Euler in 1777 and many others to the present, see [7, 1] and the references therein. Mathematicians of the time wanted to know how the k complex roots $at^k + bt^\ell + c$ depend on the choice of the coefficients a, b and c . For example, how the roots can be characterized geometrically, how many of them lie in a disk of given radius or whether two roots share the same norm. Algebraically, these questions are well understood – particularly due to P. Bohl’s results from 1908. According to Theobald, et al, [18], P. Bohl considered a fixed support $\mathcal{A} = \{0, \ell, k\} \subset \mathbb{Z}_{\geq 1}$ where k and ℓ are coprime and examined the behavior of the norms of the roots of univariate trinomials $t^k + pt^\ell + q$ where $p, q \in \mathbb{C} - \{0\}$. The author showed that the roots of $t^k + pt^\ell + q$ are parameterized using hypotrochoid curves, see [18, Theorems 3.1, 3.2] and the references therein. In addition, the geometry of roots in the complex plane is well described by P. Nekrassoff in 1887 [11] and J. Egerv ry in 1922 – 1931 (see the survey [16]).

Of all the results in the literature about this problem, we are interested in those that take the form of bounds on the magnitude of the zeros or of sectors in the complex plane containing the zeros. These stem from J. Egerv ry’s beautiful geometric description of the locations of zeros of these polynomials in the complex plane. Egerv ry proved that, the zeros of trinomials can be interpreted as the equilibrium points of a force field created by unit masses that are located at the vertices of two regular concentric polygons centered at the origin in the complex plane, see [18, Theorems 3.4, 3.5] and [16] for details. Utilising the symmetry and continuity properties of this force field, the roots of a trinomial can be separated according to their argument and moduli. This explains why roots are located in the sections, which were described by P. Nekrassoff [11]. In 2012, A. Melman obtained finer results involving smaller annular sectors containing the zeros of a trinomial that take into account the magnitude of the coefficients, see [7].

Most of the approaches in the above studies could be characterised as being algebraic and/or geometric. However, most recently, in [1], D. Belki c employed some analytic tools that primarily focus upon derivations of the analytical formulae for all the roots of trinomials through series developments using the Bell polynomials and the Fox-Wright function. These are based on the fact, that all the roots of the general n th degree trinomial admit certain convenient representations in terms of the Lambert and Euler series for the asymmetric and symmetric cases of the trinomial equation, respectively. As an application of their work, these analytical solutions are numerically illustrated in the genome multiplicity corrections for survival of synchronous cell populations after irradiation, see [1] for details.

From a different view point, we consider a fixed support $\mathcal{A} = \{0, \ell, k\} \subset \mathbb{Z}_{\geq 1}$ where k and ℓ are coprime as above. Theobald, et al., [18] used amoeba theory to investigate the geometry and topology of the parameter space $\mathcal{T}_{\mathcal{A}}$ of the trinomials with support $\mathcal{A}$. They obtained the following topological characterization:- the space of trinomials with identical root norms is connected but not simply connected, and deformation retracted to a torus knot; in addition an exception occurs when certain norms coincide, leading to a different topology with fundamental group $\mathbb{Z}^2$.

Despite the above efforts, the problem of explicitly locating the zeros of a general trinomial is still unresolved. In the current paper, we take a different approach and

instead study the location of ratios of zeros of trinomials. In particular, we choose an $n \in \mathbb{Z}_{\geq 1}$, then a $z_0 \in \mathcal{Z}_{P_n(z)} - \mathcal{Z}_{A(z)}$ and study the location of ratios of the zeros of the associated trinomial $D(t; z_0) = A(z_0)t^k + B(z_0)t^\ell + 1$. We obtain the following results, namely: Theorem 1.5 and Corollary 2.9.

THEOREM 1.5. (Main result) *If $\{P_n(z)\}$ is the polynomial sequence generated by (1) and $z_0 \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$, then there exists at-least two zeros of $D(t; z_0) = A(z_0)t^k + B(z_0)t^\ell + 1$ whose ratio is real or has modulus 1.*

The remaining sections of the paper are devoted to proving Theorem 1.4 and Theorem 1.5 in reverse order.

2. Proofs.

2.1. Some lemmata and important results. Let $\mathcal{C} := \{z \in \mathbb{C} : |z| = 1\}$ and $h : X \rightarrow \mathbb{C}$, $w \mapsto \frac{(1 - w^k)^k}{(1 - w^\ell)^\ell (w^\ell - w^k)^{k-\ell}}$ where X is the domain of h .

LEMMA 2.1. *If $z \in X \cap (\mathbb{R} \cup \mathcal{C})$, then $h(z) \in \mathbb{R}$.*

Proof. The domain of h is $X = \mathbb{C} - (\mu_\ell \cup \mu_{k-\ell} \cup \{0\})$ where μ_n denotes the set of n th complex zeros of unity. Now, there are two cases to consider, namely:

(i) if $z \in X \cap \mathbb{R}$, then $z \in \mathbb{R}$, hence $h(z) \in \mathbb{R}$ as h is a quotient of two polynomials with real coefficients, i.e., $h \in \mathbb{R}(w)$.

(ii) if $z \in X \cap \mathcal{C}$, then $h(z) \in \mathbb{R}$ by [10, Lemma 2]. □

We now optimize h over $X \cap \mathbb{R}$ since its optimal solutions on $\mathbb{R}$ have some connection with Theorem 1.5. To do this, we first define the (analytic) extension

g of h at $x = 1$ as follows: $g : (X \cap \mathbb{R}) \cup \{1\} \rightarrow \mathbb{R}$, $x \mapsto \begin{cases} h(x), & x \in X \cap \mathbb{R}, \\ \frac{k^k}{\ell^\ell (k-\ell)^{k-\ell}}, & x = 1. \end{cases}$

To investigate the behaviour of g , it suffices to study g over $[-1, 1] - \{0\}$ since h satisfies a functional relation $h(q) = h(q^{-1})$.

PROPOSITION 2.2. *If*

(i) k is odd, then g has 3 stationary points with the global minimum value

$$g_{\min} = \frac{k^k}{\ell^\ell (k-\ell)^{k-\ell}} \text{ attained at each of the three stationary points.}$$

(ii) k is even, then g has 2 stationary points with a local minimum value

$$g_{\text{loc. min}} = \frac{k^k}{\ell^\ell (k-\ell)^{k-\ell}} \text{ at } q = 1 \text{ and a local maximum value } g_{\text{loc. max}} = 0 \text{ at } q = -1.$$

Proof. The optimal values of g occur at the critical points of g , i.e., either $g'(q)$ does not exist or $g'(q) = 0$. Now

$$g'(q) = \frac{-(1-q^k)^{k-1}q^{\ell-1}(k-\ell+\ell q^k-kq^\ell)((k-\ell)q^k-kq^{k-\ell}+\ell)}{(1-q^\ell)^{\ell+1}(q^\ell-q^k)^{k-\ell+1}}. \quad (2)$$

We note that, $g'(q)$ does not exist if $q \in (\mu_\ell \cup \mu_{k-\ell}) \cap \mathbb{R}$. There are two cases to consider, depending on the parity of k :

- (i) k is odd: $g'(q)$ does not exist if $q \in \{-1, 1\}$, with $\lim_{q \rightarrow -1} g(q) = \infty = \lim_{q \rightarrow 0} g(q)$

$$\text{and } g(1) = \frac{k^k}{\ell^\ell(k-\ell)^{k-\ell}}.$$

- (ii) k is even: $g'(q)$ does not exist if $q \in \{0, 1\}$. Now $\lim_{q \rightarrow 0} g(q)$ does not exist while

$$g(1) = \frac{k^k}{\ell^\ell(k-\ell)^{k-\ell}}.$$

For $g'(q) = 0$, there are also two cases to be considered depending on the parity of k :

- (i) k is odd: $g'(q) = 0$ if and only if q is a real solution to either $k-\ell+\ell q^k-kq^\ell = 0$ or $(k-\ell)q^k-kq^{k-\ell}+\ell = 0$ since $q \notin \{0, 1\}$. Since $k-\ell+\ell q^k-kq^\ell$ and $(k-\ell)q^k-kq^{k-\ell}+\ell$ are reciprocal polynomials of each other in the variable q , it suffices to solve any one of them. If $\gamma \neq 1$ is a solution to $(k-\ell)q^k-kq^{k-\ell}+\ell = 0$, then we have

$$(k-\ell)\gamma^k = -\ell + k\gamma^{k-\ell}. \quad (3)$$

Evaluating g at such a point γ gives

$$\begin{aligned} g(\gamma) &= \frac{(1-\gamma^k)^k}{(1-\gamma^\ell)^\ell(\gamma^\ell-\gamma^k)^{k-\ell}} \\ &= \frac{((k-\ell)-(k-\ell)\gamma^k)^k}{(1-\gamma^\ell)^\ell(1-\gamma^{k-\ell})^{k-\ell}\gamma^{\ell(k-\ell)}} \cdot \frac{1}{(k-\ell)^k}, && \text{by (3),} \\ &= \frac{(k-k\gamma^{k-\ell})^k}{(1-\gamma^\ell)^\ell(1-\gamma^{k-\ell})^{k-\ell}\gamma^{\ell(k-\ell)}} \cdot \frac{1}{(k-\ell)^k} \\ &= \frac{(1-\gamma^{k-\ell})^\ell}{(1-\gamma^\ell)^\ell\gamma^{\ell(k-\ell)}} \cdot \frac{1}{(k-\ell)^\ell} \cdot \frac{k^k}{(k-\ell)^{k-\ell}}, \\ &= \frac{(1-\gamma^{k-\ell})^\ell}{((k-\ell)\gamma^{k-\ell}-(k-\ell)\gamma^k)^\ell} \cdot \frac{k^k}{(k-\ell)^{k-\ell}}, && \text{by (3),} \\ &= \frac{(1-\gamma^{k-\ell})^\ell}{((k-\ell)\gamma^{k-\ell}+\ell-k\gamma^{k-\ell})^\ell} \cdot \frac{k^k}{(k-\ell)^{k-\ell}} \\ &= \frac{(1-\gamma^{k-\ell})^\ell}{(1-\gamma^{k-\ell})^\ell} \cdot \frac{k^k}{\ell^\ell(k-\ell)^{k-\ell}} = \frac{k^k}{\ell^\ell(k-\ell)^{k-\ell}} = g(1). \end{aligned}$$

Since $g(q) > 0$ for $q \in (-1, 1] - \{0\}$, the functional equation $h(q) = h(q^{-1})$ implies $g(q) > 0$ for all $q \in \mathbb{R} - \{-1, 0\}$. In addition, $\lim_{q \rightarrow \pm\infty} g(q) = \lim_{q \rightarrow \infty} h(q) = \lim_{q \rightarrow 0} h(q^{-1}) = \lim_{q \rightarrow 0} h(q) = \infty$. Therefore, $g(1)$ is a global minimum value of g .

- (ii) k is even: $g'(q) = 0$ if and only if either $q = -1$ or $q \neq 1$ is a real solution to either

$$k - \ell + \ell q^k - kq^\ell = 0 \text{ or } (k - \ell)q^k - kq^{k-\ell} + \ell = 0. \quad (4)$$

Clearly, $g(-1) = 0$ and for any $\gamma \neq 1$ which is a solution to either equation in (4), we have $g(\gamma) = g(1)$ by the calculation in part (i) above. Furthermore, since $g(q) \leq 0$ for $q \in [-1, 0)$, the functional equation $h(q) = h(q^{-1})$ implies $g(q) \leq 0$ for $q < 0$ hence 0 is a local maximum. Similarly, as $g(q) \geq g(1)$ for $q \in (0, 1]$, the functional equation $h(q) = h(q^{-1})$ implies $g(q) \geq g(1)$ for $q > 0$ hence we must have $g(1)$ as a local minimum of g .

We now determine the number of stationary points for g in each case. From Equation (2), the real roots of $g'(q)$ are a subset of the real roots of its (not in reduced form) numerator $-(1 - q^k)^{k-1}q^{\ell-1}L(q)$ where $L(q) = (k - \ell + \ell q^k - kq^\ell)((k - \ell)q^k - kq^{k-\ell} + \ell) = (k\ell - \ell^2)q^{2k} + (k^2 - k\ell)q^{2k-\ell} - (k^2 - k\ell)q^{k+\ell} + (k^2 - 2k\ell + 2\ell^2)q^k - (k^2 - k\ell)q^{k-\ell} - k\ell q^\ell + (k\ell - \ell^2)$. Since $k > \ell \geq 1$ and the domain of g is $(\mathbb{C} - (\mu_\ell \cup \mu_{k-\ell} \cup \{0\})) \cap \mathbb{R}$, although $q = 0$ is a root of the numerator of $g(q)$, it is not a zero of $g'(q)$ since g is not defined at $q = 0$, (g has a vertical asymptote there since $\lim_{q \rightarrow 0} g(q) = \infty$). Furthermore, the polynomial $L(q)$ has 4 sign changes hence at-most 4 positive real roots by Descartes' rule of signs. On the other hand, from the factorisation $L(q) = (k - \ell + \ell q^k - kq^\ell)((k - \ell)q^k - kq^{k-\ell} + \ell)$, we observe that $q = 1$ is a root for $L(q)$ of multiplicity (order) 4 hence the only positive root of $L(q)$, however the rational function $g'(q)$ has a pole at $q = 1$ of order 3, hence $g'(q)$ has a zero at $q = 1$ of multiplicity 1. Therefore, g has a stationary point at $q = 1$, with $g(1) = \frac{k^k}{\ell^\ell(k - \ell)^{k-\ell}}$.

For the negative turning points, there are two cases:

1. k is even: $L(-q) = (k\ell - \ell^2)q^{2k} - (k^2 - k\ell)q^{2k-\ell} + (k^2 - k\ell)q^{k+\ell} + (k^2 - 2k\ell + 2\ell^2)q^k + (k^2 - k\ell)q^{k-\ell} + k\ell q^\ell + (k\ell - \ell^2)$ has 2 sign changes hence at-most 2 negative roots by Descartes' rule of signs. From the factorisation $L(q) = (k - \ell + \ell q^k - kq^\ell)((k - \ell)q^k - kq^{k-\ell} + \ell)$, we observe that $q = -1$ is a root for $L(q)$ of multiplicity (or order) 2 hence the only negative root. Furthermore, $g'(q)$ has a factor $(1 - q^k)^{k-1}$ hence $q = -1$ is a zero of multiplicity at $k - 1 + 2$. Since k is even, g does not change concavity. In particular, since g has a local maximum at $q = -1$, it follows that g is concave downwards for $q < 0$. and the stationary point $q = -1$ of g is unique for $q < 0$.
2. k is odd: if ℓ is odd, then $L(-q)$ has 2 sign changes hence at-most 2 negative roots by Descartes' rule of signs. Since k and ℓ are both odd, the factor $k - \ell + \ell q^k + kq^\ell$ of $L(-q)$ must have all real roots $q < 0$. In addition, q^{-1} will be a negative root of the other factor $((k - \ell)q^k + kq^{k-\ell} + \ell)$ of $L(q)$ by

the symmetry. This shows that g has exactly two stationary points when k and ℓ are both odd. On the other hand, if ℓ is even, then $L(-q)$ has 4 sign changes hence at-most 4 negative roots by Descartes' rule of signs. Since k is odd and ℓ is even, the factor $(k - \ell + \ell q^k - k q^\ell)$ of $L(-q)$ must have a real root $q < 0$, clearly different from -1 . In addition, q^{-1} will be a root to the other factor of $L(q)$. This shows that g has at-least two stationary points. Since $g(q)$ takes on the same value $g(1)$ at all its stationary points, it follows that g can not have more than 2, since any other stationary point will happen away from the value $g(1)$, contradicting $g(\gamma) = g(1)$ for any stationary point γ (as in the first part of the proof of Proposition 2.2). Since $g(q) > 0$ for all $q < 0$, and $\lim_{h \rightarrow \pm\infty} g(q) = \lim_{h \rightarrow 0} g(q^{-1}) = \lim_{q \rightarrow 0} h(q) = \infty = \lim_{q \rightarrow -1} g(q)$, it follows that g is always concave upwards. $\square$

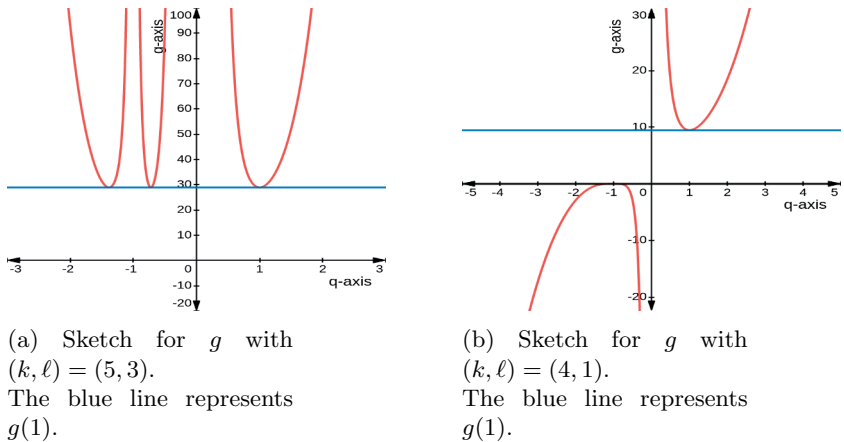


Figure 1

DEFINITION 2.3. Let $n \in \mathbb{Z}_{\geq 1}$. The q -discriminant of an n th degree polynomial $f(y)$ with zeros $y_1, \dots, y_n$ and leading coefficient a_n is; (we use the convention that an empty product evaluates to 1)

$$\Delta_q(f(y)) = a_n^{2n-2} q^{\frac{n(n-1)}{2}} \prod_{1 \leq i < j \leq n} \left(y_i^2 + y_j^2 - \left(q + \frac{1}{q} \right) y_i y_j \right).$$

If $q = 1$, then $\Delta_q(f(y)) = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (y_i - y_j)^2 =: \Delta(f(y))$, the classical discriminant of $f(y)$.

THEOREM 2.4. ([10, Theorem 2]) Let k, ℓ be coprime integers with $k > \ell \geq 1$ and $D(t; z) = A(z)t^k + B(z)t^\ell + 1$ where $A(z), B(z) \in \mathbb{C}[z] - \{0\}$. For any $z_0 \in \mathbb{C}$ with

$A(z_0)B(z_0) \neq 0$, the q -discriminant $\Delta_q(D(t; z_0))$ of the trinomial $D(t; z_0)$ is

$$(-1)^{\frac{k(k+3)}{2}} (A(z_0))^{k-1} (B(z_0))^{\ell-1} \frac{(1-q^k)^k}{(1-q)^k} \left(1 - \frac{(1-q^\ell)^\ell (q^\ell - q^k)^{k-\ell}}{(q^k - 1)^k} \frac{(B(z_0))^k}{(A(z_0))^\ell} \right). \quad (5)$$

REMARK 2.5. We have the following remarks.

1. Vanishing of the q -discriminant for $D(t; z_0)$ at $q = 1$ implies existence of repeated zeros for $D(t; z_0)$.

If $q_0 \in \mathbb{C}$ is a ratio of zeros of $D(t; z_0)$, then $\Delta_{q_0}(D(t; z_0)) = 0$, i.e., $(-1)^k \frac{B^k(z_0)}{A^\ell(z_0)} = \frac{(1-q_0^k)^k}{(1-q_0^\ell)^\ell (q_0^\ell - q_0^k)^{k-\ell}} =: h(q_0)$. In addition, $h(q) = h(q_0)$ for any other ratio q of distinct zeros of $D(t; z_0)$.

2. If $z_0 \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$, then $(-1)^k \frac{B^k(z_0)}{A^\ell(z_0)} \in \mathbb{R}$ by [2, Theorem 1.1].

REMARK 2.6. Let $\{P_n(z)\}$ be the sequence of polynomials generated by (1), $z_0 \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$, $\alpha = (-1)^k \frac{B^k(z_0)}{A^\ell(z_0)}$, $\beta = \frac{k^k}{\ell^\ell (k-\ell)^{k-\ell}}$ and $\Omega_{k,\ell}(\alpha)$ be the number of real ratios of distinct zeros of $D(t; z_0)$. By Proposition 2.2:

$$(i) \text{ If } k \text{ is odd, then } \Omega_{k,\ell}(\alpha) = \begin{cases} 0, & \alpha < \beta, \\ 2, & \alpha = \beta, \\ 6, & \alpha > \beta. \end{cases} \text{ This follows because, if } \alpha < \beta, \text{ then}$$

$\alpha = h(q) = g(q)$ has no real solutions since $g(q) > \beta$ for $q \in \mathbb{R}$. For $\alpha = \beta$, there are 3 real ratios one of which is $q = 1$ arising from non-distinct zeros of $D(t; z_0)$. For $\alpha > \beta$, we have $\alpha = h(q) = g(q)$ has 6 distinct real solutions, 4 of which are negative.

$$(ii) \text{ If } k \text{ is even, then } \Omega_{k,\ell}(\alpha) = \begin{cases} 2, & \alpha < 0, \text{ or } \alpha > \beta, \\ 1, & \alpha = 0, \\ 0, & 0 < \alpha \leq \beta. \end{cases} \text{ This follows because, if}$$

$\alpha < \beta$, then $\alpha = h(q) = g(q)$ has no real solutions since $g(q) > \beta$ for $q \in \mathbb{R}$. For $\alpha = \beta$, there is actually a real ratio $q = 1$ (but it is not of distinct zeros of $D(t; z_0)$). For $\alpha > \beta$, we have $\alpha = h(q) = g(q)$ has 6 distinct real solutions, 4 of which are negative.

Remark 2.6 is illustrated using Figure 1 for some specific choices of k and ℓ . Any other choice of k and ℓ will give similar behaviour. The β is related to the separability threshold $\sigma(k, \ell)$ in [7, Definition 2.3] as follows: $\beta = (\sigma(k, \ell))^k$.

LEMMA 2.7. If $\{P_n(z)\}$ is the sequence generated by (1), $z_0 \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$ and $q \in \mathbb{C} - (\mathbb{R} \cup \mathbb{C})$ is a ratio of distinct zeros of $D(t; z_0)$, then there is a ratio $q' \neq q$ of distinct zeros of $D(t; z_0)$ with $q' \in \mathbb{R} \cup \mathbb{C}$ and $h(q') = h(q) = (-1)^k \frac{B^k(z_0)}{A^\ell(z_0)}$.

Proof. Suppose that $q \in \mathbb{C} - (\mathbb{R} \cup \mathcal{C})$ is a ratio of distinct zeros of $D(t; z_0)$. We show existence of another ratio $q' \in \mathbb{R} \cup \mathcal{C}$ of distinct zeros of $D(t; z_0)$. To this end, we let $A(z_0) = |A(z_0)|e^{i\beta\pi}$ and $B(z_0) = |B(z_0)|e^{i\theta\pi}$, where θ and β are rational numbers. Since $k, \ell \in \mathbb{Z}_{\geq 1}$ are coprime, and $h(q) = (-1)^k \frac{B^k(z_0)}{A^\ell(z_0)} \in \mathbb{R}$, we have $\frac{k}{\ell} = \frac{\beta}{\theta} \pm \frac{\gamma}{\ell\theta}$ where $\gamma \in \{-1, 0, 1\}$. The transformation $Y = e^{i\lambda\pi}t$ where $\lambda = \frac{\beta - \theta + \gamma}{k - \ell}$ transforms $D(t; z_0)$ into $\overline{D}(Y; z_0) := \pm |A(z_0)|Y^k \pm |B(z_0)|Y^\ell + 1 \in \mathbb{R}[Y]$, (note, there are four possible polynomials). Via the transformation $Y = e^{i\lambda\pi}t$, the ratios of (distinct) zeros of the polynomials $D(t; z_0)$ and $\overline{D}(Y; z_0)$ are the same. So it is enough to study the ratios of distinct zeros of $\overline{D}(Y; z_0)$.

- (i) When $k \geq 2$ is odd, then there must exist a real zero of $\overline{D}(Y; z_0)$. If it is the only zero, then there exists a complex conjugate pair $(y_1, \overline{y_1})$ of zeros of $\overline{D}(Y; z_0)$ whose ratio is in $\mathcal{C}$. In the event that there are more than one real zeros of $\overline{D}(Y; z_0)$, then we have a real ratio.
- (ii) When k is even, either there are no real zeros or there must be at-least two real zeros of $\overline{D}(Y; z_0)$. In the latter case, there must exist a real ratio while in the former case, the zeros must occur in complex conjugate pairs, hence there must exist a ratio q in $\mathcal{C} - \{1\}$. $\square$

THEOREM 2.8. *If $\{P_n(z)\}$ is the polynomial sequence generated by (1), $z_0 \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$ and q is a ratio of distinct zeros of $D(t; z_0)$, then $h(q) \in \mathbb{R}$ if and only if either $q \in (\mathbb{R} \cup \mathcal{C}) \cap X$ or there is another ratio $q' \in (\mathbb{R} \cup \mathcal{C}) \cap X$ of distinct zeros of $D(t; z_0)$.*

Proof. We now show that if $q \in (\mathbb{R} \cup \mathcal{C}) \cap X$ is a ratio of distinct zeros of $D(t; z_0)$, then $h(q) \in \mathbb{R}$. There are two cases: Either $q \in \mathbb{R} \cap X$, in which case $h(q) \in \mathbb{R}$ since $h(z) \in \mathbb{R}(z)$ or $q \in \mathcal{C} \cap X$, in which case we have $h(q) \in \mathbb{R}$ by Lemma 2.1. On the other hand, if $q \notin X - (\mathbb{R} \cup \mathcal{C}) \cap X$ is a ratio of distinct zeros of $D(t; z_0)$, then Lemma 2.7 implies existence of another ratio q' of distinct zeros of $D(t; z_0)$ such that $q' \in X \cap (\mathbb{R} \cup \mathcal{C})$, which again implies that $h(q) = h(q') \in \mathbb{R}$.

($\Rightarrow$) Suppose that $h(q) \in \mathbb{R}$ where q is a ratio of distinct zeros of $D(t; z_0)$. Since $q \neq 1$, Remark 2.5 implies that,

$$h(q) = \frac{(1 - q^k)^k}{(1 - q^\ell)^\ell (q^\ell - q^k)^{k-\ell}} = (-1)^k \frac{B^k(z_0)}{A^\ell(z_0)} = \alpha.$$

We consider the polynomial function

$$f(w) := (1 - w^k)^k - \alpha(1 - w^\ell)^\ell (w^\ell - w^k)^{k-\ell}. \quad (6)$$

Since $\alpha \in \mathbb{R}[w]$, it follows that $f \in \mathbb{R}[w]$ and has degree k^2 . Moreover, $f(w) = (1 - w)^k H(w)$ where

$$H(w) = \left(\frac{1 - w^k}{1 - w} \right)^k - \alpha \left(\frac{1 - w^\ell}{1 - w} \right)^\ell \left(\frac{1 - w^{k-\ell}}{1 - w} \right)^{k-\ell} w^{\ell(k-\ell)}.$$

It is clear that H extends to $\hat{H} \in \mathbb{R}[w]$ with $\hat{H}(1) := \lim_{w \rightarrow 1} H(w)$.

Clearly, $\hat{H}(1) = \lim_{w \rightarrow 1} H(w) = k^k - \alpha \ell^\ell (k - \ell)^{k-\ell} = 0$ if and only if $\alpha = \frac{k^k}{\ell^\ell (k - \ell)^{k-\ell}} = g(1)$ which is forbidden (since $q \neq 1$, we consider distinct zeros of $D(t; z_0)$). Therefore, 1 is a zero of f of multiplicity k , hence $\deg(\hat{H}) = k^2 - k \in 2\mathbb{Z}_{\geq 1}$. Since the zeros of $\hat{H}$ and H are the same (up to multiplicity), without loss of generality, we can take $H = \hat{H} \in \mathbb{R}[w]$. Now $\mathcal{W}_H \subset \mathcal{W}_f$ where $\mathcal{W}_H$ is the set of ratios of distinct zeros of $D(t; z_0)$ while $\mathcal{W}_f$ is the set of ratios of zeros of $D(t; z_0)$, not necessarily distinct. So if $q \in \mathcal{W}_H$, then so is $\bar{q}$ and q^{-1} hence $\bar{q}^{-1}$ as $H(w) \in \mathbb{R}[w]$. Hence, $h(q) = h(\bar{q}) = h(q^{-1}) = h(\bar{q}^{-1}) = \alpha$. Consequently, we have the following possibilities among the zeros of H .

- (i) $q = \bar{q}$ implies that $q \in \mathbb{R} - \{1\}$.
- (ii) $q = q^{-1}$ implies that $q^2 = 1$ hence $q = -1 \in \mathbb{R}$ as $q \neq +1$.
- (iii) $q = \bar{q}^{-1}$ implies that $q\bar{q} = 1$ hence $|q| = 1$, thence $q \in \mathcal{C} - (\mu_\ell \cup \mu_{k-\ell})$.
- (iv) Suppose that q is a ratio of distinct zeros of $D(t; z_0)$, but none of the conditions (i)–(iii) holds. The conclusion that there is another ratio q' of distinct zeros of $D(t; z_0)$ for which $q' \in (\mathbb{R} \cup \mathcal{C}) \cap X$ follows from Lemma 2.7. $\square$

Let us finally settle the main result of the present paper.

Proof of Theorem 1.5. Let $z_0 \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$, and $t_i := t_i(z_0)$ for $i = 1, 2, \dots, k$ be the zeros of the trinomial $D(t; z_0) = A(z_0)t^k + B(z_0)t^\ell + 1$. There are two cases we shall consider; namely, repeated and distinct zeros of $D(t; z_0)$.

1. *Case 1:* (Repeated zeros). The classical discriminant of $D(t; z_0)$ vanishes. In this case, at least two of the zeros of D are equal and nonzero. Therefore, at least one of the ratios of zeros of D is $1 \in \mathbb{R}$ and on $\mathcal{C}$.
2. *Case 2:* (Distinct zeros). In this case, the classical discriminant of $D(t; z_0)$ does not vanish but its q -discriminant vanishes at any ratio q of distinct zeros of $D(t; z_0)$. Let q_0 be a ratio of distinct zeros of $D(t; z_0)$, i.e., $(q_0 \neq 1)$. Now, $h(q_0) = \frac{(1 - q_0^k)^k}{(1 - q_0^\ell)^\ell (q_0^\ell - q_0^k)^{k-\ell}} = (-1)^k \frac{B^k(z_0)}{A^\ell(z_0)} \in \mathbb{R}$ by [2, Theorem 1.1]. Since $h(q_0) \in \mathbb{R}$, we have either $q_0 \in (\mathbb{R} \cup \mathcal{C}) \cap X$ or there is another ratio $q' \in (\mathbb{R} \cup \mathcal{C}) \cap X$ of distinct zeros of $D(t; z_0)$, by Theorem 2.8. $\square$

Let $z_0 \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$, and $t_i := t_i(z_0)$ for $i = 1, 2, \dots, k$ be the zeros of the trinomial $D(t; z_0) = A(z_0)t^k + B(z_0)t^\ell + 1$ and $\mathcal{R}$ be the set of all ratios of zeros of the trinomial $D(t; z_0)$. We say $D(t; z_0)$ has null-collinearity property if: for even values of k , exactly two of the ratios of distinct roots of $D(t; z_0)$ lie on a straight line through the origin in the complex plane, otherwise (for odd values of

k), there are exactly three such ratios, see Remark 2.6. This property indicates a special geometric structure in the distribution of the roots, where certain ratios of the zeros are constrained to align in a linear fashion, reflecting the symmetry in the configuration of the distinct roots of $D(t; z_0)$.

COROLLARY 2.9. *If $\{P_n(z)\}$ is the polynomial sequence generated by the recurrence (1) and $z_0 \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$, then $D(t; z_0)$ has at-least two equimodular zeros or the set of zeros exhibits null-collinearity.*

Proof. For integers $k > \ell \geq 1$ such that k and ℓ are coprime, Theorem 1.5 tells us that, there are either at-least two zeros of $D(t; z_0)$ whose ratios are real or have modulus 1. Now, if the ratio of zeros has modulus 1, then the two roots are equimodular. Now, suppose that at-least two ratios of distinct roots are real. There are two cases to be considered:

1. k is odd. In this case, there are 6 ratios of distinct roots depending on α and β as in Remark 2.6. This in turn corresponds to 3 (not necessarily) distinct roots of $D(t; z_0)$. Since the ratios are roots to the equation $h(x) = \alpha$, the analysis on the turning points in Proposition 2.2 indicates that four of the ratios are negative.
2. k is even. In this case, there are 2 ratios of distinct roots depending on α and β as in Remark 2.6. This in turn corresponds to 2 (not necessarily) distinct roots of $D(t; z_0)$. Since the ratios are roots to the equation $h(x) = \alpha$, the analysis on the turning points in Proposition 2.2 indicates that the two ratios have the same sign. $\square$

If $F(t) := t^k + at^\ell - 1$ where $a \in \mathbb{C} - \{0\}$, then the reality of $(-1)^k \frac{B^k(z_0)}{A^\ell(z_0)}$ in Remark 2.5(3), corresponds to the reality of $(-a)^k$, [7]. Corollary 2.9 reveals a property exhibited by the roots of $F(t)$ if $(-a)^k$ is real, i.e., $F(t)$ has at-least two equimodular zeros and / or exactly three null-collinear zeros for k odd, or exactly two null-collinear zeros for k even.

From Corollary 2.9, we note that the roots of $D(t; z_0) = 0$ are not arbitrary; they exhibit certain regularities due to the nature of the equation. The ratios of the roots help reveal how these zeros are positioned in the complex plane. The null-collinearity condition essentially captures how these roots are geometrically related to one another. When k is even, the alignment of two ratios of the roots on a straight line through the origin suggests a balanced symmetry about the origin, while for odd k , the condition extends to a more intricate structure involving three such ratios. This alignment provides insights into the nature of the polynomial roots, suggesting that they exhibit a specific symmetry which may be intrinsic to the recurrence relations governing the sequence of polynomials.

It is important to note that null-collinearity is intrinsic to $D(t : z_0)$ (forced by the condition $\frac{B^k(z_0)}{A^\ell(z_0)} \in \mathbb{R}$) and invariant under the symmetries of the involution action : $\mathbb{C} - \{0\} \rightarrow \mathbb{C} - \{0\}$, $z \mapsto z^{-1}$ and circle (1-dimensional compact torus)

n	$z_{n,j} \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$	$A(z_{n,j}) = 0$	is $(-1)^k \frac{B^k(z_{n,j})}{A^\ell(z_{n,j})} \in \mathbb{R}$?	$D(t; z_{n,j})$	$q_{n,s}^* = q_{n,s} \in \mathbb{C} - \{0, 1\}$
17	$(1 - 2\sqrt{2})\mathbf{i}$	no	yes, 0	$D(t; z_{17,0})$	$-\mathbf{0.8090} \pm \mathbf{0.5878i}$ $\mathbf{0.3090} \pm \mathbf{0.9511i}$
23	$-0.6109 - 2.2046\mathbf{i}$	no	yes, 0.71429	$D(t; z_{23,0})$	$0.5553 \pm 1.2966\mathbf{i}$ $0.0372 \pm 1.2114\mathbf{i}$ $-0.9024 \pm 0.8090\mathbf{i}$ $\mathbf{0.6444} \pm \mathbf{0.7647i}$ $0.0253 \pm 0.8247\mathbf{i}$ $0.2791 \pm 0.6517\mathbf{i}$ $-\mathbf{0.6900} \pm \mathbf{0.7238i}$ $-1.0553 \pm 0.4908\mathbf{i}$ $-0.7791 \pm 0.3623\mathbf{i}$ $-0.6144 \pm 0.5508\mathbf{i}$
56	$-0.2985 + 3.1410\mathbf{i}$	no	yes, 0.01943	$D(t; z_{56,0})$	$0.4198 \pm 1.1044\mathbf{i}$ $0.1927 \pm 1.0854\mathbf{i}$ $0.1586 \pm 0.8932\mathbf{i}$ $0.3007 \pm 0.7911\mathbf{i}$ $\mathbf{0.4731} \pm \mathbf{0.8810i}$ $-0.8651 \pm 0.6833\mathbf{i}$ $-0.9198 \pm 0.5501\mathbf{i}$ $-\mathbf{0.7475} \pm \mathbf{0.6643i}$ $-0.8007 \pm 0.4789\mathbf{i}$ $-0.7119 \pm 0.5623\mathbf{i}$
56	$1.9038 + 1.6907\mathbf{i}$	no	yes, 99.18923	$D(t; z_{56,1})$	$1.0694 \pm 2.0772\mathbf{i}$ 1.8190 0.5497 $0.1960 \pm 0.3806\mathbf{i}$ $-0.9605 \pm 1.8655\mathbf{i}$ $-0.5280 \pm 1.0256\mathbf{i}$ -1.1134 $-\mathbf{0.5810} \pm \mathbf{0.8139i}$ -0.8981 -0.4937 $-0.2182 \pm 0.4237\mathbf{i}$ $-0.3968 \pm 0.7708\mathbf{i}$ -2.0254

Table 1: Data with $k = 5, \ell = 3, A(z) = \mathbf{i}z^3 + z + 3\mathbf{i}, B(z) = z^2 - 2\mathbf{i}z + 7$.

actions on the complex plane. The involution swaps pairs of roots, while the compact torus action rotates the roots in the plane (as observed in the proof of Lemma 2.7). Despite their differences, both actions preserve the geometric relationships of the roots, ensuring that the null-collinearity condition is maintained. At certain points, the involution and compact torus actions coincide, highlighting a deeper connection between their respective transformations.

For a demonstration for Theorem 1.5 and Corollary 2.9, we considered $A(z) = \mathbf{i}z^3 + z + 3\mathbf{i}, B(z) = z^2 - 2\mathbf{i}z + 7$, with $k = 5$ and $\ell = 3$. We numerically computed some few terms of the corresponding sequence $\{P_n(z)\}$, in particular $\{P_{17}(z), P_{23}(z), P_{56}(z)\}$, and their corresponding zeros. Out of these zeros, we selected a few denoted by $z_{n,j}$ satisfying $A(z_{n,j}) \neq 0$, computed $D(t; z_{n,j})$ as well as ratios of distinct zeros of $D(t; z_{n,j})$ and summarised the results in Table 1.

In Table 1, we have

$$\begin{aligned} D(t; z_{17,0}) &:= ((25 + 4i) - (22 + 2i)\sqrt{2})t^5 + 1, \\ D(t; z_{23,0}) &:= -(8.85784 - 9.47542i)t^5 - (1.89645 - 3.91559i)t^3 + 1, \\ D(t; z_{56,0}) &:= (29.8505 + 14.9508i)t^5 + (3.50521 - 1.27839i)t^3 + 1, \\ D(t; z_{56,1}) &:= -(11.6461 + 4.7345i)t^5 + (11.1473 + 2.62973i)t^3 + 1. \end{aligned}$$

All computations were done using Mathematica [23] and/or SageMath software [15] on a 64-bit machine with 5-cores and intel microprocessor with 4.0 GB of RAM and processor speed 2.4GHz. In Figure 2, we illustrate Theorem 1.5 by showing the location of ratios of distinct zeros of the polynomials $D(t; z_{n,j})$ corresponding to the choice of zero $z_{n,j}$ of each of the polynomials $P_{17}(z)$, $P_{23}(z)$ and $P_{56}(z)$ in Table 1. In Figure 3, we illustrate Corollary 2.9.

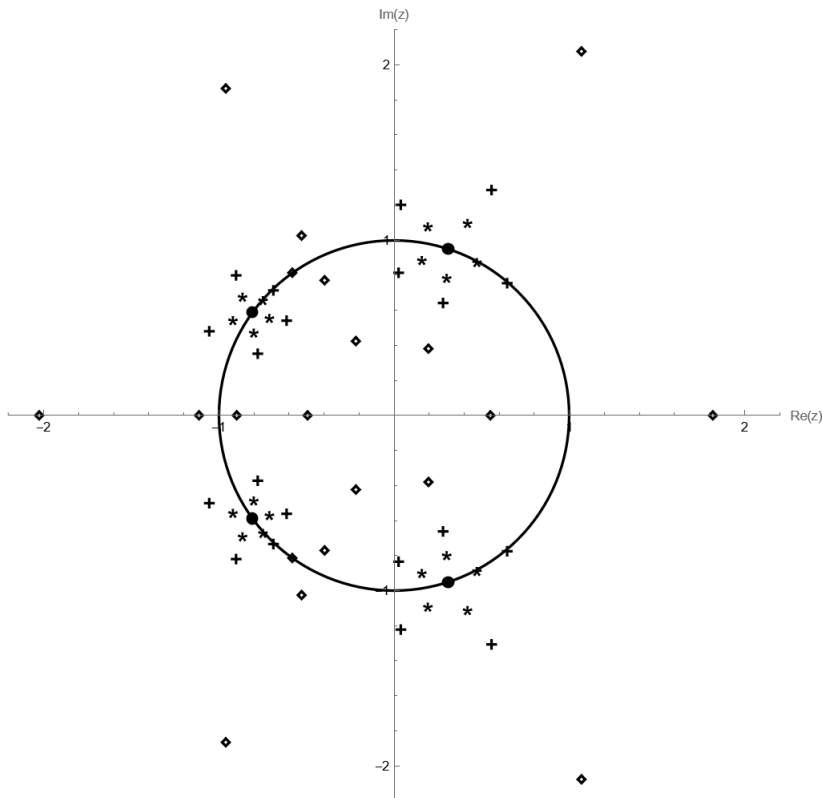


Figure 2: The markers ‘•’, ‘+’, ‘*’ and ‘◊’ correspond to the ratios of distinct zeros of $D(t; z_{17,0})$, $D(t; z_{23,0})$, $D(t; z_{56,0})$, and $D(t; z_{56,1})$ respectively. The green curve is $\mathcal{C}$, (unit circle centered at the origin).

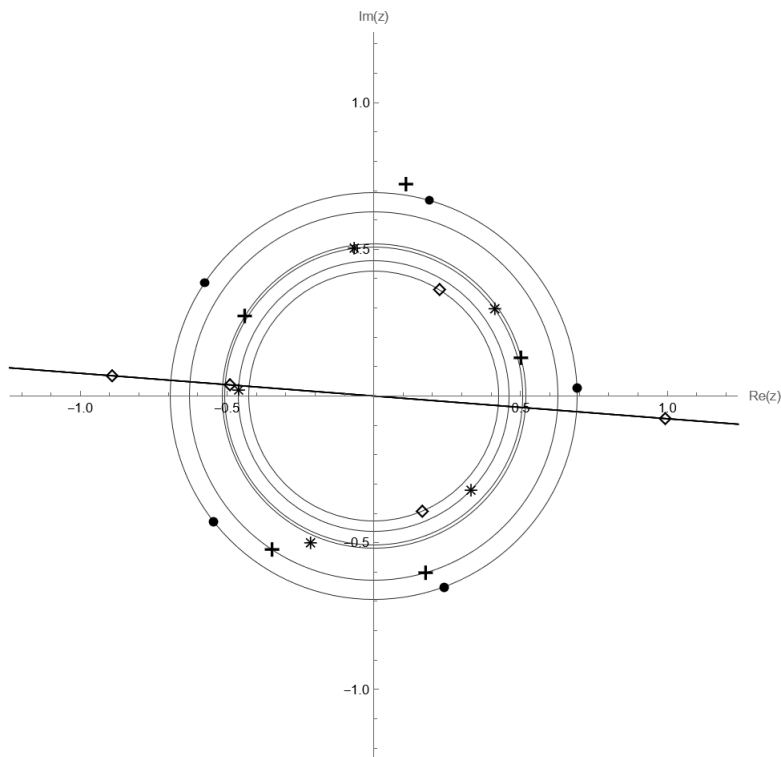


Figure 3: The markers ‘•’, ‘+’, ‘*’ and ‘◊’ correspond to the zeros of $D(t; z_{17,0})$, $D(t; z_{23,0})$, $D(t; z_{56,0})$, and $D(t; z_{56,1})$ respectively. Similar markers on the same circle represent equimodular zeros of a given $D(t; z_0)$. The three ‘◊’ markers on a line via the origin are null-collinear.

REMARK 2.10. We have some remarks.

1. For each of the four markers, i.e., ‘•’, ‘+’, ‘*’ and ‘◊’ in Figure 2, there are at least two markers on $\mathcal{C}$ which correspond to the highlighted points (boldface) in Table 1.
2. For each non-real complex number $q_{n,j}^*$ lying off $\mathcal{C}$, Theorem 1.5 guarantees existence of another ratio $q_{n,j}^{*'} of distinct zeros of $D(t; z_{n,j})$ such that $q_{n,j}^* \neq q_{n,j}^{*'}$ and either $q_{n,j}^{*'}$ is real or lies on $\mathcal{C}$.$
3. The zero $z_{17,0} = (1 - 2\sqrt{2})\mathbf{i}$ is a special zero for $P_{17}(z)$ since $B(z_{17,0}) = 0$. If $B(z_0) = 0$, then Theorem 1.5 trivially holds since the zeros of $D(t; z_0) = A(z_0)t^k + 1$ (recall $A(z_0) \neq 0$) are equimodular (of the form $t_j = \left| \frac{1}{A(z_0)} \right| e^{\frac{2\pi i j}{k}}$ for $j = 0, 1, \dots, k-1$). Clearly, any ratio of two distinct zeros of $D(t; z_0)$ lie on $\mathcal{C}$, and each occurs with multiplicity k . This explains why in Figure 2, all the yellow colour points lie on $\mathcal{C}$ and there are four of them instead of 20.

- The case for $n = 17$ and $z_0 = (1 - 2\sqrt{2})\mathbf{i}$ in Table 1 yields the corresponding $D(t; z_0)$, (although not a trinomial) yields all ratios on the unit circle implying the roots are equimodular as expected. As such it yields a borderline point in the parameter space $T_{0,3,5}$ on a piece that is deformation retractable to a torus knot. All the other examples given in the Table 1 yield ratios some of which are unimodular (although not all) signifying that the z_0 's yield point in the parameter space $T_{0,3,5}$ on pieces that are connected with fundamental group $\mathbb{Z}^2$, as in [18].

The ratios of distinct zeros of $D(t; z_0)$ may all be real only when $k^2 - k \leq 6$, i.e., $k \in \{2, 3\}$. For example, for $k = 2$, we can consider $A(z) = z^2$, $B(z) = z^2 - 2z - 5$ and $n = 1$ which yields $\alpha = 0$, thence $q = -1$ as the only (real) ratio of distinct zeros. For $k = 3$, a simple algebraic manipulation shows that $D(t; z_0)$, must have a pair of repeated zeros, implying $\alpha = \frac{27}{4}$ and hence exactly two real ratios of distinct roots of $D(t; z_0)$. Numerical experiments suggest that this might be a limiting case, with for example $A(z) = 2z$, $B(z) = 3z$, see Figure 4 for the results.

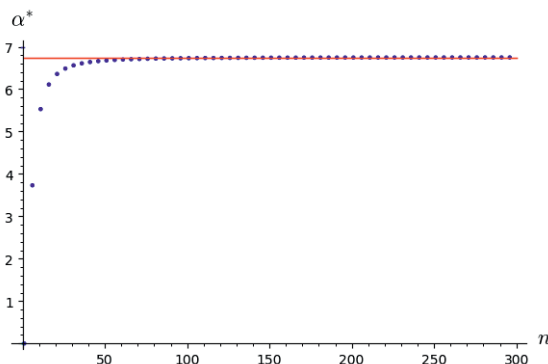


Figure 4: Plot of $\alpha^* := \max \left\{ \frac{(-B(z_0))^3}{A(z_0)} : z_0 \in \mathcal{Z}_{P_n(z)} - \mathcal{Z}_{A(z)} \right\}$ against $n \in \{1 + 5\lambda : \lambda = 0, 1, 2, \dots, 59\}$ for $k = 3$, $\ell = 1$, $A(z) = 2z$, and $B(z) = 3z$.

Now for $k \geq 2$, Corollary 2.9 guarantees existence of either equimodularity or null-collinearity on the zeros of the trinomial $t^k + B(z_0)t^\ell + A(z_0)$, where $z_0 \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$. It follows by [18, Theorem 5.6 and 5.8], that for each $z_0 \in \mathcal{Z}_{\{P_n(z)\}} - \mathcal{Z}_{A(z)}$, the constants $A(z_0)$ and $B(z_0)$ either belong to the connected but not simply connected space with fundamental group isomorphic to $\mathbb{Z}$ if some of the zeros of $D(t; z_0)$ are not equimodular, or to a connected space with fundamental group isomorphic to $\mathbb{Z}^2$ if all the zeros of $D(t; z_0)$ are equimodular.

2.2. The real algebraic curve $\Gamma_{k,\ell}$. In this subsection, we shall show that the inequalities stated in Theorem 1.4 are related to the zeros of $\{P_n(z)\}$. For completeness of the exposition, we consider the sequence $\{P_n(z)\}$ defined by the recurrence in Equation (1).

For the triple (ℓ, k, n) , where $\ell, k, n \in \mathbb{Z}_{\geq 1}$ (k and ℓ are fixed coprime integers with $k > \ell \geq 1$ used in the definition of the recurrence (1)), we define $S := S_{\ell, k, n} = \{(i, j) \in \mathbb{Z}_{\geq 0}^2 : i\ell + jk = n\}$. Clearly, S is a finite set say with s elements. This follows from the fact that, the linear Diophantine equation $x\ell + ky = n$ is a straight line with negative gradient, and the sought for solutions lie in the first quadrant of the xy -plane. In general, a linear Diophantine equation $x\ell + ky = n$ (with $\gcd(k, \ell) = 1$) has integer solutions of the form $(x, y) = (x_0 + ku, y_0 - \ell u)$ where $x_0, y_0, u \in \mathbb{Z}$. Moreover, we can choose an x_0 and y_0 so that we parameterise S as follows: $S = \{(i_u, j_u) := (x_0 + ku, y_0 - \ell u) \in \mathbb{Z}_{\geq 0}^2 : u = 1, \dots, s\}$.

To the triple (ℓ, k, n) , we associate the polynomial

$$G_{\ell, k, n}(\tau) := \sum_{u=1}^s \binom{i_u + j_u}{i_u} \tau^{u-1}$$

in τ . This is the generating function for the number of north-east lattice paths starting from the origin to a lattice point (i_u, j_u) lying on the linear Diophantine equation $x\ell + ky = n$ in the first quadrant of the xy -plane. Such polynomials $G_{\ell, k, n}(\tau)$ are real rooted since their coefficients form a Pólya frequency sequence. In addition, all the roots of $G_{\ell, k, n}(\tau)$ are negative (as all its coefficients are non-negative), see [24, Conjecture 1] and [2] for details. In [2, Lemma 2], Bögvad et al., showed that

$$P_n(z) = \pm B^{i_1}(z) A^{j_1}(z) G_{\ell, k, n} \left((-1)^{(k-\ell)} \frac{B^k(z)}{A^\ell(z)} \right).$$

Using the real-rootedness property of $G_{\ell, k, n}(\tau)$, Bögvad et al., proved that, if $z_0 \in \mathcal{Z}_{P_n(z)} - \mathcal{Z}_{A(z)B(z)}$, then

$$G_{\ell, k, n} \left((-1)^{(k-\ell)} \frac{B^k(z_0)}{A^\ell(z_0)} \right) = 0,$$

hence $\operatorname{Im} \left((-1)^{(k-\ell)} \frac{B^k(z_0)}{A^\ell(z_0)} \right) = 0$, as in Theorem 1.2.

We pointed out in Remark 1.3 that, if $\ell = 1$, then $\operatorname{Re} \left((-1)^{(k-\ell)} \frac{B^k(z_0)}{A^\ell(z_0)} \right)$

is bounded in absolute value by $\frac{k^k}{(k-1)^{k-1}}$, but the case for $\ell > 1$ had remained out of reach. We now address this case by showing that, if $\ell > 1$, then $\operatorname{Re} \left((-1)^{(k-\ell)} \frac{B^k(z_0)}{A^\ell(z_0)} \right)$ can be arbitrarily large. To obtain this result using brute force, one would have to consider all the cases of n and observe the behaviour of roots of $G_{\ell, k, n}(\tau)$. However, this is cumbersome and yields complicated expressions for one to deal with. A clever way is to use simpler cases in which the linear Diophantine equation has a solution on the axes, i.e., those in which either $n \equiv 0 \pmod{\ell}$ or $n \equiv 0 \pmod{k}$ and the simplest as $n \equiv 0 \pmod{k\ell}$. For the remaining part of the arguments/computations, we consider zeros of the polynomials $P_n(z)$ where $n \equiv 0 \pmod{k\ell}$.

LEMMA 2.11. *If $n = \gamma k\ell$ for some $\gamma \in \mathbb{Z}_{\geq 1}$, then $G_{\ell,k,n}(\tau)$ is monic with constant term 1 and sum of roots $-\left(\frac{n}{\ell} - k + \ell\right)$.*

Proof. If $n \equiv 0 \pmod{k\ell}$, then among the non-negative solutions of the linear Diophantine equations $x\ell + ky = n$ are the pairs $\left(0, \frac{n}{k}\right)$ and $\left(\frac{n}{\ell}, 0\right)$. Since $S = \{(i_u, j_u) := (x_0 + ku, y_0 - \ell u) \in \mathbb{Z}_{\geq 0}^2 : u = 1, \dots, s\}$, the first solutions corresponds to $u = 1$ while the last to $u = s$. In particular, if $u = s$, then $j_s = 0$ and $\binom{i_s + j_s}{i_s} = \binom{i_s}{i_s} = 1$ which implies that $G_{\ell,k,n}(\tau)$ is monic. Similarly, if $u = 1$, then $i_1 = 0$ and $\binom{i_1 + j_1}{i_1} = \binom{j_1}{0} = 1$ implying that $G_{\ell,k,n}(\tau)$ has constant term 1. Since $G_{\ell,k,n}(\tau)$ is monic, the sum of its roots is $-\binom{i_{s-1} + j_{s-1}}{i_{s-1}}$, the negative of the coefficient of τ^{s-2} in $G_{\ell,k,n}(\tau)$. However, $\binom{i_{s-1} + j_{s-1}}{i_{s-1}} = \binom{\frac{n}{\ell} - k + \ell}{\frac{n}{\ell} - k}$ which follows from the choice of $u = s - 1$ giving $j_{s-1} = \ell$ and $i_{s-1} = \frac{n}{\ell} - k$. $\square$

We determine s explicitly. Let $N(a, b, n)$ be the number of non-negative integer solutions of a linear Diophantine equation $ax + by = c$ with $\gcd(a, b) = 1$. In [22, Theorem], Tripathi showed that

$$N(a, b, n) = \frac{n + a'(n)a + b'(n)b}{ab} - 1,$$

where $a'(n) \equiv -na^{-1} \pmod{b}$ with $1 \leq a'(n) \leq b$ and $b'(n) \equiv -nb^{-1} \pmod{a}$, with $1 \leq b'(n) \leq a$.

LEMMA 2.12. *If $\gcd(k, \ell) = 1$ and $n = \gamma k\ell$ for some $\gamma \in \mathbb{Z}_{\geq 1}$, then $N(\ell, k, n) = \gamma + 1$.*

Proof. Suppose $a = \ell$, $b = k$. Since $\gcd(k, \ell) = 1$ and $n = \gamma k\ell$ for some $\gamma \in \mathbb{Z}_{\geq 1}$, it follows from [22, Theorem] that $1 \leq a'(n) \equiv 0 \pmod{k} \leq k$, hence $a'(n) = k$ and $1 \leq b'(n) \equiv 0 \pmod{\ell} \leq \ell$, hence $b'(n) = \ell$. Therefore,

$$N(\ell, k, n) = \frac{\gamma k\ell + a'(n)a + b'(n)b}{\ell k} - 1 = \frac{\gamma k\ell + k\ell + \ell k}{k\ell} - 1 = \gamma + 1. \quad \square$$

LEMMA 2.13. *If $\ell > 1$, $\gcd(k, \ell) = 1$, $z_0 \in \mathcal{Z}_{P_n(z)} - \mathcal{Z}_{A(z)}$ and $\mathcal{Z}_{B(z)} - \mathcal{Z}_{A(z)} \neq \emptyset$, then $-\infty < \operatorname{Re} \left((-1)^{(k-\ell)} \frac{B^k(z_0)}{A^\ell(z_0)} \right) \leq 0$.*

Proof. Firstly, from the recurrence (1), we observe that, if $n = \ell$, then $P_\ell(z) = -B(z)$. Therefore, if $z_0 \in \mathcal{Z}_{P_\ell(z)} - \mathcal{Z}_{A(z)}$, then $B(z_0) = 0$ hence

$$\operatorname{Re} \left((-1)^{(k-\ell)} \frac{B^k(z_0)}{A^\ell(z_0)} \right) = 0.$$

Let $\tau_1, \dots, \tau_{s-1}$ be the roots of $G_{\ell,k,n}(\tau)$ with $\tau^* = \min\{\tau_i\}_{i=1}^{s-1}$, i.e., $|\tau^*|$ is the largest root in absolute value. Since $G_{\ell,k,n}(\tau)$ is monic, it follows that $\sum_{i=1}^{s-1} \tau_i = -\binom{i_{s-1} + j_{s-1}}{i_{s-1}}$. Since all the roots of $G_{\ell,k,n}(\tau)$ are negative, $\frac{1}{s-1} \sum_{i=1}^{s-1} \tau_i \geq \frac{1}{s-1} \sum_{i=1}^{s-1} \tau^* = \tau^*$ hence

$$|\tau^*| = -\tau^* \geq -\frac{1}{s-1} \sum_{i=1}^{s-1} \tau_i = \frac{1}{s-1} \binom{i_{s-1} + j_{s-1}}{i_{s-1}}.$$

If $n = \gamma k \ell$, for some $\gamma \in \mathbb{Z}_{\geq 1}$, then $s = \gamma + 1$ by Lemma 2.12, hence

$$\frac{1}{s-1} \binom{i_{s-1} + j_{s-1}}{i_{s-1}} = \frac{1}{\gamma} \binom{\frac{n}{\ell} - k + \ell}{\frac{n}{\ell} - k} = \frac{1}{\gamma} \binom{(\gamma-1)k + \ell}{(\gamma-1)k} \approx \frac{(\gamma-1)^\ell k^\ell}{\ell! \gamma}.$$

As γ hence n tends to ∞ , it follows that $\frac{1}{s-1} \binom{i_{s-1} + j_{s-1}}{i_{s-1}}$, (hence $|\tau^*|$) becomes arbitrarily large. $\square$

Proof of Theorem 1.4. If $\ell > 1$, and $\mathcal{Z}_{B(z)} \subset \mathcal{Z}_{A(z)}$, then for any $z_0 \in \mathcal{Z}_{P_n(z)} - \mathcal{Z}_{A(z)}$, we have $B(z_0) \neq 0$, hence $\operatorname{Re} \left((-1)^{(k-\ell)} \frac{B^k(z_0)}{A^\ell(z_0)} \right) \neq 0$. Therefore, the complete proof Theorem 1.4 follows from Theorem 1.2 + Lemma 2.13. $\square$

REMARK 2.14. In the situation $(k, \ell) = (k, 1)$, we have

$$\frac{1}{s-1} \binom{i_{s-1} + j_{s-1}}{i_{s-1}} = \frac{1}{\gamma} \binom{(\gamma-1)k + 1}{(\gamma-1)k} = \frac{(\gamma-1)k + 1}{\gamma}$$

which implies that the average values of the roots of $G_{k,\ell,n}(\tau)$ are bounded by k for all $n \equiv 0 \pmod{k\ell}$. This is in agreement with the results obtained earlier in [19, Theorems 1, 3, 5] respectively and [20, Theorem 1] where it was proved that any $z_0 \in \mathcal{Z}_{P_n(z)} - \mathcal{Z}_{A(z)}$ must satisfy the condition that $0 \leq \operatorname{Re} \left((-1)^k \frac{B^k(z_0)}{A(z_0)} \right) \leq \frac{k^k}{(k-1)^{k-1}}$. This is because, as n tends to ∞ so does γ and hence the expression $\frac{(\gamma-1)k + 1}{\gamma}$ tends to $k < \frac{k^k}{(k-1)^{k-1}}$ for any $k \in \mathbb{Z}_{\geq 2}$.

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Appendix A. SageMath code

```

1 #Sage code for generating the data in Table 1, and graphs in Figure 1 and Figure
  2.
2 #Given coprime integers k and l such that k>l>=1, arbitrary complex polynomials
  A and B such that degree of A(z)B(z) is atleast 1 and positive integer n,
  the code computes the polynomial Pn(z) in polynomial sequence {Pn(z)}
  generated by the recurrence in equation (1). Furthermore, it computes the
  zeros of Pn(z). Using a zero z0 of one's choice (such that A(z0) is nonzero
  ), the code numerically computes the trinomial D(t;z0) = A(z0)t^k+B(z0)t^l
  +1, its zeros, ratios of zeros of D(t;z0) and then plots both the zeros and
  ratios of zeros of D(t;z0) on two separate graphs.
3 #To minimise the numerical errors (mostly when computing the ratios of zeros of
  D(t;z0)), we include a precision parameter, and for comparison, we have
  included another way of computing these ratios using alpha = (-1)^k(B(z0))^
  k/(A(z0))^l and polynomial f, equation (6) in the proof of Theorem 2.8.
4
5 #variables
6 z,t,q=var('z,t,q') #names of variables of different polynomials used in the code
7
8 #parameters (needed to reproduce the graphs in the paper)
9 k, l = 5, 3 # coprime integers k and l such that k>l>=1
10 A, B = i*z^3+z+3*i, z^2 -2*i*z+7 # complex polynomials A and B such that degree
  of A(z)B(z) is >= 1
11 precision = 50 # precision paramater, change to 100, 150 or 200 for accuracy (at
  the expense of speed)
12 markers = ['o','+', '*', 'd'] # for different polynomials.
13
14 #data used in the code to reproduce the graphs, includes choice of n and label
  for zero zj of Pn(z).
15 list1=[(17,1),(23,2),(56, 9), (56,35)] # (n,j)=(23,2) corresponds to P23(z) and
  z2 = -0.6109 - 2.2046i
16 #one must choose a label j and therefore a zero zj such that A(zj) is not zero.
17
18 #functions

```

```

19 def recursive(n,k,l): # inputs parameters n,k and l as described above and
    computes Pn(z) recursively.
20     if 1-k<=n<0:
21         return 0 # returns polynomials in initial conditions.
22     elif n==0:
23         return 1 # returns polynomial P0(z).
24     else:
25         return -B*recursive(n-1,k,l)-A*recursive(n-k,k,l) # returns polynomial
    Pn(z) for n>=1.
26 def wzeros(w, precision): # returns complex zeros of a function w(z) computed
    numerically.
27     if w.degree(z)>=1:
28         wroots=w.roots(ring=ComplexField(precision))
29         return wroots # complex zeros of w(z) computed to given precision.
30     elif w==0: # for this choice of w, some parts of the code may not work well.
31         return w, "is a zero polynomial."
32     else: # for this choice of w, some parts of the code may not work well.
33         return w, "is a nonzero constant polynomial, therefore, it has no zeros."
34 def polyfforratios(q,x): # polynomial in the proof of Theorem 2.8 (its zeros
    will be used as controls)
35     return (1-q^k)^k-x*(1-q^l)^l*(q^l-q^k)^(k-l)
36
37 #our plots (initialisation with reference unit circles colored green)
38 s1=plot(circle((0,0),1,aspect_ratio=1,rgbcolor=(0,1,0),linestyle='-'))
39 s11=plot(circle((0,0),1,aspect_ratio=1,rgbcolor=(0,1,0),linestyle='-'))
40 count=0 # counter as we run through list1, will help in plotting all graphs on
    the same plot.
41
42 for j in list1: #program to run through list1 to generate the graphs
43     n, choiceofzero=j[0], j[1] # e.g., (23,2)
44     w = (recursive(n,k,l)).expand() # expands the recursive polynomial as a
    polynomial in z
45     print("Our choice of Pn(z) is: ", w) # e.g., P23(z)
46     zerosofw = wzeros(w,precision) # e.g., this list contains zeros of P23(z)
47     print("The zeros of our choice of Pn(z) with their multiplicities are:")
48     for counter, s in enumerate(zerosofw): # e.g., label, zero of P23(z),
    multiplicities
49         print(" %d: " % counter, zerosofw[counter][0], zerosofw[counter][1])
50     yourchoice = zerosofw[choiceofzero][0] # e.g., z2=-0.6109 - 2.2046i, for P23
    (z)
51     w0 = A(z=yourchoice)*t^k+B(z=yourchoice)*t^l+1 # we already checked that A(
    zj) is nonzero,
52     # e.g., D(t;z2)=- (8.85784 - 9.47542i)t^5 - (1.89645 - 3.91559i)t^3 + 1.
53     print("Your choice of zero is: ", yourchoice, "and the corresponding D(t,z0)
    is: ", w0)
54     zerosofw0 = w0.roots(ring=ComplexField(precision)) # seros of D(t;z0)
55     print("The zeros of D(t,z0) with their multiplicities are:")
56     for counter, s in enumerate(zerosofw0): # e.g., label, zero of D(t,z2),
    multiplicities
57         print(" %d: " % counter, zerosofw0[counter][0], zerosofw0[counter][1])
58     alpha = (-1)^k*(B(z=yourchoice))^k/(A(z=yourchoice))^l # alpha parameter to
    be used in f
59     print(alpha) # e.g., alpha=
60     w1 = polyfforratios(q,alpha) # polynomial f with alpha, see (6) in the proof
    of Theorem 2.8
61     ratiosofzerosofD = w1.roots(ring=ComplexField(precision))
62     print("The theoretical ratios of (not necessarily distinct) zeros of D(t,z0)
    :")
63     for counter, s in enumerate(ratiosofzerosofD): # e.g., label, zero of f,
    multiplicities
64         print(" %d: " % counter, ratiosofzerosofD[counter][0], ratiosofzerosofD[
    counter][1])
65     zerosofDnum = [b[1][0] for b in enumerate(zerosofw0)] # e.g., zeros of D(t,
    z2) computed numerically
66     cart_zerosofDnum = cartesian_product([zerosofDnum,zerosofDnum]) # e.g.,
    cartesian product.
67     ratiosofzerosofDnum = [b[0]/b[1] for b in cart_zerosofDnum] # e.g., ratios
    of zeros of D(t,z2)

```

```

68     print("The numerical ratios of (not necessarily distinct) zeros of D(t,z0):"
69         )
70     for counter, s in enumerate(ratiosofzerosofDnum):# label, ratios of zeros of
71         D(t,z0), multiplicities
72         print(" %d: "% counter, ratiosofzerosofDnum[counter])
73     points_ratiosofzerosofD=[d[0] for d in ratiosofzerosofD] # ratios of zeros
74     of D(t,z0) to be plotted
75     s1=s1+points(points_ratiosofzerosofD, marker=markers[count],size=30) #
76     append plot of ratios of zeros of D(t,z0)
77     points_zerosofD=[d[0] for d in zerosofw0] # zeros of D(t,z0) to be plotted
78     below
79     s11=s11+points(points_zerosofD, marker=markers[count],size=30) # append plot
80     of ratios of zeros of D(t,z0)
81     for j in points_zerosofD: #appending circles centered at origin containing
82         each zero of D(t,z0)
83         s11=s11+plot(circle((0,0),j.abs(),aspect_ratio=1,thickness=0.3,rgbcolor
84             =(0,0,0),linestyle='-'))
85         count=count+1
86     s1.axes_labels([r"$\mathrm{Re}(z)$",r"$\mathrm{Im}(z)$"])
87     save(s1,'/tmp/figure1.png')
88     s11.axes_labels([r"$\mathrm{Re}(z)$",r"$\mathrm{Im}(z)$"])
89     save(s11,'/tmp/figure2.png')
90     os.system('display /tmp/figure2.png')
91     os.system('display /tmp/figure2.png')
92     show(s1,fontsize=4.5) # for jupyter notebook, sagemath cell or cocalc
93     interpreters.
94     show(s11,fontsize=4.5) # for jupyter notebook, sagemath cell or cocalc
95     interpreters.

```